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F-SPACES UNIVERSAL WITH RESPECT TO LINEAR CODIMENSION

WESLEY E. TERRY

ABSTRACT. Rolewicż raised the question in [5] as to whether there existed a separable F-space X_0 such that any other separable F-space Y is the image of X_0 under a continuous linear operator. This can be equivalently phrased as the question [5, Problem II.4.3, p. 47]: Does there exist a separable F-space universal for all separable F-spaces with respect to linear codimension? Theorem 1 proves the existence of such a separable F-space. Theorem 2 generalizes this idea to larger cardinals.

1. **Introduction.** A metric linear space X is called an *F-space* if X has a complete metric ρ with the property that $\rho(x+z, y+z) = \rho(x, y)$ for all x, y, $z \in X$. Such a metric, ρ, is said to be *invariant*. By a theorem of Kakutani [5, Theorem I.1.1, p. 12], every metric linear space has an invariant metric. By a theorem of Klee [5, I.4.3, p. 24] if X is a topologically complete metric linear space then an invariant metric for the topology on X is a complete metric on X.

Let Λ be a set of cardinality $\aleph$. The space $l_1(\aleph)$ is defined to be the set of all real functions $r = \{r_\lambda\}$ defined on the set Λ with at most a countable number of nonzero elements and with $\Sigma_\lambda |r_\lambda| < \infty$. The norm on $l_1(\aleph)$ is $\|r\| = \Sigma_\lambda |r_\lambda|$.

Schauder proved the following theorem:

THEOREM (*Schauder, see Banach-Mazur* [1, *p.* 111] *and Klee* [4, *Proposition* 2.1]). *Every Banach space B, with the weight of B less than or equal to $\aleph$, is a linear image of $l_1(\aleph)$.*

Theorems 1 and 2 are motivated by this result. Theorem 3 proves that the universal spaces of Theorems 1 and 2 are homeomorphic to $l_1(\aleph)$ for appropriate cardinal $\aleph$.

2. Proofs of Theorems.

THEOREM 1. *There is a separable F-space universal with respect to linear codimension for all separable F-spaces.*

PROOF. Let $H(I)$ be the space of all homeomorphisms of the unit interval $I = [0, 1]$ onto itself that are the identity on the endpoints and satisfy the following property:

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If $a, b, c \in [0, 1]$ with $a \leqslant b + c$, then $F(a) \leqslant F(b) + F(c)$ for $F \in H(I)$.The metric on $H(I)$ is the supremum metric.

Pick a countable dense subset of $H(I)$. Call it $\{g_i\}_{i=1}^{\infty}$. Pick the rationals in $(0, 1]$ and enumerate them as $\{p_i\}_{i=1}^{\infty}$. Let $\{f_i\}_{i=1}^{\infty}$ be an enumeration of the countable number of elements resulting from replacing each g_i in $\{g_i\}_{i=1}^{\infty}$ by the countable number of elements $\{p_j g_i\}_{j=1}^{\infty}$.

Define h to be the homeomorphism

$$h: [0, \infty) \to [0, 1), \qquad h(r) = r/(1 + r).$$

Then each f_i induces an invariant, strictly monotone metric on $\mathbf{R} = (-\infty, \infty)$ defined by

$$d_i(r, s) = f_i(h(|r - s|)).$$

d_i is a metric since $f_i = p_j g_k$ for some $g_k \in H(I)$ and $h(a) \leqslant h(b) + h(c)$ if $a \leqslant b + c, a, b, c \in [0, \infty)$. $d_i(t, 0) < d_i(s, 0)$ if $0 \leqslant t < s$, since both h and f_i are strictly monotone. Finally,

$$\begin{aligned} d_i(t + s, r + s) &= f_i\big(h(|t + s - (r + s)|)\big) \\ &= f_i\big(h(|t - r|)\big) = d_i(t, r) \end{aligned}$$

for all $t, r, s \in \mathbf{R}$. Thus d_i is an invariant metric.

Now, let $\prod_{j=1}^{\infty} \mathbf{R}_j$ be the countable cartesian product of copies of $\mathbf{R}$. Define

$$\sum_{l_1} (\mathbf{R}, d_j) = \left\{ \{x_j\} \in \prod_{j=1}^{\infty} \mathbf{R}_j \middle| \{d_j(x_j, 0)\}_{j=1}^{\infty} \in l_1 \right\}.$$

Here $l_1 = l_1(\aleph_0)$. Then $\Sigma_{l_1}(\mathbf{R}, d_j)$ is an F-space under coordinatewise addition and scalar multiplication and under the invariant metric $\sum_{j=1}^{\infty} d_j(x_j, y_j) = \rho(\{x_j\}, \{y_j\})$. Given this invariant metric ρ,

$$|\{x_j\}|_{l_1} = \sum_{j=1}^{\infty} d_j(x_j, 0)$$

is called the associated F-norm. [Given an invariant metric ρ on an F-space, the associated F-norm is defined to be $| \; | = \rho(\cdot, 0)$.]

We claim that $(\Sigma_{l_1}(\mathbf{R}, d_j), | \; |_{l_1})$ is the required universal F-space. $\Sigma_{l_1}(\mathbf{R}, d_j)$ is clearly separable. Given a separable F-space $(M, | \; |)$, we may assume that the F-norm is not only invariant but also strictly monotone on rays from the origin and bounded by one. (See Eidelheit and Mazur [3].) Pick a countable dense subset of M from $M \setminus \{0\}$, and call this collection $\{x_i\}_{i=1}^{\infty}$.

Look at the lines $L_i = \{rx_i | r \in \mathbf{R}\}$, $i = 1, 2, \ldots$. Then, for each i, $(L_i, | \; |_i)$ is an F-space where $| \; |_i$ is $| \; |$ restricted to L_i. Let

$$s_i = \sup_{s \in (0, \infty)} |sx_i|_i \leqslant 1.$$

Then $| \; |_i$ takes on values in $[0, s_i)$ since $|-x| = |x|$ by definition of an F-norm and $| \; |$ is strictly monotone so that $| \; |_i$ cannot take on the supremum.

Define F_i: $[0, s_i] \to [0, s_i]$ by

$$F_i(r) = \begin{cases} |(r/(s_i - r))x_i|_i, & r \neq s_i, \\ s_i, & r = s_i. \end{cases}$$

Then F_i is a homeomorphism of $[0, s_i]$ onto itself since $|\ |$ is strictly monotone. Thus $F(s_i \cdot)/s_i \in H(I)$ since $|\ |_i$ satisfies the triangle inequality and is strictly monotone. Also, $F_i(s_i \cdot)$ is a homeomorphism of $[0, 1]$ onto $[0, s_i]$.

Now, for $F_1(s_1 \cdot)$ pick an element in $\{f_j\}_{j=1}^{\infty}$ such that

$$\sup_{r \in [0,1]} |F_1(s_1 r) - f_{\sigma(1)}(r)| < \tfrac{1}{2}.$$

Then inductively select an element in $\{f_j\}_{j=1}^{\infty} \setminus \{f_{\sigma(1)}, \ldots, f_{\sigma(i-1)}\}$ for $F_i(s_i \cdot)$ such that $\sup_{r \in [0,1]}|F_i(s_i r) - f_{\sigma(i)}(r)| < 1/2^i$. This can be done since rational scalar multiples of the $\{g_i\}_{i=1}^{\infty}$ have been included in $\{f_i\}_{i=1}^{\infty}$.

Define a linear operator A from $\Sigma_{l_1}(\mathbf{R}, d_j)$ to M by defining it on the Schauder basis $\{\delta_j\}_{j=1}^{\infty}$ where

$$\delta_j = \begin{cases} 0, & i \neq j, \\ 1, & i = j, \end{cases} \qquad \delta_j \in \sum_{l_1} (\mathbf{R}, d_j),$$

$$A\colon \sum_{l_1} (\mathbf{R}, d_j) \to M, \quad A(\delta_j) = \begin{cases} 0, & j \neq \sigma(i) \text{ for any } i, \\ x_j, & j = \sigma(i) \text{ for some } i. \end{cases}$$

We claim that A is a continuous linear surjection.

A is into since $F_i(s_i r) < 1/2^i + f_{\sigma(i)}(r)$ and, therefore, $|rx_i|_i < 1/2^i + d_{\sigma(i)}(r, 0)$ for each i and any $r \in [0, 1]$. Hence

$$\begin{aligned} \sum_{i=1}^{\infty} |r_{\sigma(i)} x_i|_i &< \sum_{i=1}^{\infty} \left[\frac{1}{2^i} + d_{\sigma(i)}(r_{\sigma(i)}, 0) \right] \\ &= 1 + \sum_{i=1}^{\infty} d_{\sigma(i)}(r_{\sigma(i)}, 0) \\ &< \infty \quad \text{for } \{r_j\} \in \sum_{l_1} (\mathbf{R}, d_j). \end{aligned}$$

Note that the induced F-norm on $\mathbf{R}$ induced by $F_i(s_i \cdot)$ is $d(r, t) = F_i(s_i h(|r - t|))$. Hence

$$d(r, t) = F_i\big(s_i h(|r - t|)\big) = |(r - t)x_i|_i.$$

To see that A is continuous, let $\{r_j\}, \{r_j^n\} \in \Sigma_{l_1}(\mathbf{R}, d_j)$, $n = 1, 2, \ldots$, and suppose $\{r_j^n\} \to \{r_j\}$. Then look at the indices corresponding to $\sigma(i)$, $i = 1, 2, \ldots$. All other coordinates go to zero under A. Then

$$\sum_{i=1}^{\infty} d_{\sigma(i)}(r_{\sigma(i)}^n, r_{\sigma(i)}) \leqslant \sum_{j=1}^{\infty} d_j(r_j^n, r_j).$$

Now, given $\varepsilon > 0$, choose N_0 so that $\Sigma_{N_0+1}^{\infty} 1/2^n < \varepsilon/4$. Next, find each index of the form $\sigma(i)$ such that $\sigma(i) \leqslant N_0$. Call these indices $k_1 = \sigma(i_1), \ldots, k_t = \sigma(i_t)$. Then, given $\varepsilon/2N_0$, pick a δ where $0 < \delta < \varepsilon/4$ such that $F_{i_j}(s_{i_j} r) < \varepsilon/2N_0$ whenever $d_{k_j}(r, 0) < \delta$, $j = 1, 2, \ldots, t$. Now, given $\delta > 0$, pick $N_1 \geqslant N_0$ such that $\Sigma_{j=1}^{\infty}(r_j^n, r_j) < \delta$ whenever $n \geqslant N_1$. Then for $n \geqslant N_1$,

$$\begin{aligned}
\left|A(\{r_j^n\}) - A(\{r_j\})\right| &= \left|\sum_{i=1}^{\infty} r_{\sigma(i)}^n x_i - \sum_{i=1}^{\infty} r_{\sigma(i)} x_i\right| \\
&\leqslant \sum_{i=1}^{\infty} \left|(r_{\sigma(i)}^n - r_{\sigma(i)})x_i\right| \\
&\leqslant \sum_{j=1}^{t} \left|(r_{k_j}^n - r_{k_j})x_{i_j}\right| + \sum_{\sigma(i) > N_0+1} \left|(r_{\sigma(i)}^n - r_{\sigma(i)})x_i\right| \\
&< N_0\left(\frac{\varepsilon}{2N_0}\right) + \sum_{\sigma(i) > N_0+1} \left[d_{\sigma(i)}(r_{\sigma(i)}^n, r_{\sigma(i)}) + \frac{1}{2^i}\right] \\
&< \varepsilon/2 + \varepsilon/4 + \varepsilon/4 = \varepsilon.
\end{aligned}$$

To see that A is surjective, let $y \in M$. Again, look at $\{x_i\}_{i=1}^{\infty}$, the countable dense set in M. Pick $r_{\beta(1)}x_{\beta(1)}$ such that $|y - r_{\beta(1)}x_{\beta(1)}| < \frac{1}{2}$. Next, since $\{x_i\}_{i=1}^{\infty} \setminus \{x_{\beta(1)}\}$ is still dense in M, pick $r_{\beta(2)}x_{\beta(2)}$ so that $|(y - r_{\beta(1)}x_{\beta(1)}) - r_{\beta(2)}x_{\beta(2)}| < \frac{1}{4}$. Inductively, pick $r_{\beta(n)}x_{\beta(n)}$ where $x_{\beta(n)}$ is chosen from

$$\{x_i\}_{i=1}^{\infty} \setminus \{x_{\beta(1)}, \ldots, x_{\beta(n-1)}\}$$

so that

$$\left|(y - r_{\beta(1)}x_{\beta(1)} - \cdots - r_{\beta(n-1)}x_{\beta(n-1)}) - r_{\beta(n)}x_{\beta(n)}\right| < 1/2^n.$$

Then

$$|r_{\beta(1)}x_{\beta(1)}| \leqslant |y - r_{\beta(1)}x_{\beta(1)}| + |y| < \tfrac{1}{2} + |y| < \tfrac{1}{2} + 1,$$

$$\begin{aligned}
|r_{\beta(2)}x_{\beta(2)}| &\leqslant |(y - r_{\beta(1)}x_{\beta(1)}) - r_{\beta(2)}x_{\beta(2)}| + |y - r_{\beta(1)}x_{\beta(1)}| \\
&< \tfrac{1}{4} + \tfrac{1}{2}
\end{aligned}$$

$$\vdots$$

$$\begin{aligned}
|r_{\beta(n)}x_{\beta(n)}| &\leqslant \left|(y - r_{\beta(1)}x_{\beta(1)} - \cdots - r_{\beta(n-1)}x_{\beta(n-1)}) - r_{\beta(n)}x_{\beta(n)}\right| \\
&\quad + \left|(y - r_{\beta(1)}x_{\beta(1)} - \cdots - r_{\beta(n-2)}x_{\beta(n-2)}) - r_{\beta(n-1)}x_{\beta(n-1)}\right| \\
&< 1/2^n + 1/2^{n-1} \\
&\vdots
\end{aligned}$$

Now, $\Sigma_{k=1}^{\infty} r_{\beta(k)}x_{\beta(k)} \in M$ since

$$\left|\sum_{k=1}^{\infty} r_{\beta(k)}x_{\beta(k)}\right| \leqslant \sum_{k=1}^{\infty} |r_{\beta(k)}x_{\beta(k)}| < |y| + 2\sum_{k=1}^{\infty} \frac{1}{2^k} = |y| + 2.$$

Select $\{r_j\} \in \Sigma_{l_1}(\mathbf{R}, d_j)$ by

$$r_j = \begin{cases} r_{\beta(k)} & \text{if } j = \sigma(\beta(k)) \text{ for some } k, \\ 0 & \text{if } j \neq \sigma(\beta(k)) \text{ for any } k. \end{cases}$$

To see that $\{r_j\} \in \Sigma_{l_1}(\mathbf{R}, d_j)$ note that

$$\sup_{r \in [0,1]} |F_{\beta(k)}(S_{\beta(k)}r) - f_{\sigma(\beta(k))}(r)| < \left(\frac{1}{2}\right)^{\beta(k)}.$$

Thus, since $d_{\sigma(\beta(k))}(r_{\beta(k)}, 0) = f_{\sigma(\beta(k))}(h(r_{\beta(k)}))$,

$$\begin{aligned} d_{\sigma(\beta(k))}(r_{\beta(k)}, 0) &\leqslant (1/2)^{\beta(k)} + F_{\beta(k)}(s_{\beta(k)}h(r_{\beta(k)})) \\ &= (1/2)^{\beta(k)} + |r_{\beta(k)}x_{\beta(k)}| < (1/2)^{\beta(k)} + 1/2^k + 1/2^{k-1}. \end{aligned}$$

Therefore,

$$\begin{aligned} \sum_{j=1}^{\infty} d_j(r_j, 0) &= \sum_{k=1}^{\infty} d_{\sigma(\beta(k))}(r_{\beta(k)}, 0) \\ &\leqslant \sum_{k=1}^{\infty} \left[\left(\frac{1}{2}\right)^{\beta(k)} + \frac{1}{2^k} + \frac{1}{2^{k-1}} \right] \leqslant 1 + 3 < \infty. \end{aligned}$$

Clearly, $A(\{r_j\}) = \Sigma_{k=1}^{\infty} r_{\beta(k)}x_{\beta(k)} = y$. Hence A is surjective.

Thus A has all the properties required, and $\Sigma_{l_1}(\mathbf{R}, d_j)$ satisfies the conditions of the theorem. □

THEOREM 2. *Given an infinite cardinal number $\aleph \geqslant 2^{\aleph_0}$, there is an F-space of weight $\aleph$ universal with respect to linear codimension for all F-spaces of weight less than or equal to $\aleph$.*

PROOF. Let $H(I)$ be as in Theorem 1. Then the cardinality of $H(I)$ is $2^{\aleph_0}$. Let $\{g_t\}_{t \in \mathbf{R}}$ be an enumeration of the elements of $H(I)$. Also, the cardinality of $(0, 1]$ is $2^{\aleph_0}$ as well. Let $\{f_t\}_{t \in \mathbf{R}}$ be a new enumeration containing $2^{\aleph_0}$ elements obtained by replacing each g_t by the $2^{\aleph_0}$ elements $\{rg_t\}_{r \in (0,1]}$.

Then each f_t induces an invariant, strictly monotone metric on $\mathbf{R}$ defined by $d_t(r, s) = f_t(h(|r - s|))$ just as in Theorem 1. Note that d_t is bounded by one. This gives all possible metrics on $\mathbf{R}$ with these properties.

Now, let $\Pi_{\lambda \in \Lambda}\mathbf{R}_\lambda$ be the cartesian product of $\aleph$ copies of $\mathbf{R}$. Here, the cardinality of Λ is $\aleph$. Define

$$\sum_{l_1(\aleph)} (\mathbf{R}, d_t) = \left\{ \{x_{t\lambda}\} \in \prod_{t \in \mathbf{R}} \left(\prod_{\lambda \in \Lambda} \mathbf{R}_\lambda \right) \middle| \{d_t(x_{t\lambda}, 0)\} \in l_1(\aleph) \right\}$$

where (t, λ) is suitably re-indexed to give a correspondence with the $\aleph$ coordinates of $l_1(\aleph)$. Then $\Sigma_{l_1(\aleph)}(\mathbf{R}, d_t)$ is an F-space under coordinates addition and scalar multiplication, and under the F-norm $|\{x_{t\lambda}\}|_{l_1} = \Sigma_{t\lambda} d_t(x_{t\lambda}, 0)$. This can be shown to be the required universal space by going through the same process as in Theorem 1. However, when you get to the part where you pick $f_{\sigma(t)}$ for $F_t(s_t \cdot)$, you can pick $f_{\sigma(t)}$ equal to $F_t(s_t \cdot)$ instead of just near $F_t(s_t \cdot)$. Therefore, you have the restricted metric $|\ |_t$ from $|\ |$ as the chosen induced metric. This, in fact, makes it easier to show that the linear map A is into and continuous. Since each metric is repeated $\aleph$ times, you can still insure that you choose different directions from the origin in constructing A and in showing that A is onto. Thus, the theorem follows. □

THEOREM 3. *$\Sigma_{l_1(\aleph)}(\mathbf{R}, d_t)$ is homeomorphic to $l_1(\aleph)$.*

PROOF. Define

$$H\colon \sum_{l_1(\aleph)} (\mathbf{R}, d_t) \to \sum_{l_1(\aleph)} \left(\mathbf{R}, \frac{s_t\|\ \|}{1+\|\ \|}\right) \quad \text{by } h(\{r_\alpha\}) = \{q_\alpha r_\alpha\}$$

where each q_α is chosen so that when $r_\alpha \neq 0$, q_α is the unique positive real number such that

$$d_t(q_\alpha r_\alpha, 0) = s_t\|q_\alpha r_\alpha\| / (1 + \|q_\alpha r_\alpha\|).$$

H sends a zero coordinate to the corresponding zero coordinate. Here, $s_t = \sup_{r\in(0,\infty)} d_t(r, 0)$ and $\|\cdot\|$ is the absolute value on $\mathbf{R}$. A unique such q_α exists for each α when $r_\alpha \neq 0$ since d_t and $s_t\|\cdot\|/(1 + \|\cdot\|)$ are both strictly monotone on rays from the origin. H is thus a homeomorphism.

Now, for the separable case, let $\{p_i\}_{i=1}^\infty$ be the enumeration of the rationals in (0, 1] used in Theorem 1. Then, for each i, $M_i = \Sigma_{l_1}(\mathbf{R}, p_i\|\ \|/(1 + \|\ \|))$ is homeomorphic to l_1 by the identity map. The argument is similar to that of Lemma 6 of [6]. But then, by re-indexing the coordinates of

$$\sum_{l_1} (\mathbf{R}, s_t\|\ \|/(1 + \|\ \|)),$$

we have

$$\sum_{l_1}\left(\mathbf{R}, \frac{s_t\|\ \|}{1+\|\ \|}\right) = \sum_{l_1}\left\{M_i, \sum_{k=1}^{\infty}\left[\frac{p_i\|\ \|}{1+\|\ \|}\right]_k\right\}.$$

Finally, $\Sigma_{l_1}(M_i, \Sigma_{k=1}^\infty[p_i\|\ \|/(1 + \|\ \|)]_k)$ is homeomorphic to l_1 by Theorem 3 of [6] and the fact that l_1 is homeomorphic to $\Pi_{i=1}^\infty(l_1)_i$. (Bessaga proved in [2] that $l_1(\aleph)$ is homeomorphic to $\Pi_{i=1}^\infty(l_1(\aleph))_i$.)

For the nonseparable case, let $\mathbb{S}_i = (1/2^i, 1/2^{i-1}]$ for $i = 1, 2, \ldots$. For each i define

$$M_i = \left\{\{r_{s\lambda}\} \in \prod_{s\in\mathbb{S}_i}\left(\prod_{\lambda\in\Lambda}\mathbf{R}_\lambda\right)_s \,\middle|\, \sum_{s\in\mathbb{S}_i}\sum_{\lambda\in\Lambda}\frac{s\|r_{s\lambda}\|}{1+\|r_{s\lambda}\|} < \infty\right\}$$

with the obvious metric. The argument now proceeds as in the separable case. Each M_i is homeomorphic to $l_1(\aleph)$ and this implies $\Sigma_{l_1(\aleph)}(\mathbf{R}, s_t\|\ \|/(1 + \|\ \|))$ is homeomorphic to $l_1(\aleph)$. □

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DEPARTMENT OF MATHEMATICAL SCIENCES, NEW MEXICO STATE UNIVERSITY, LAS CRUCES, NEW MEXICO 88003