PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 63, Number 1, March 1977

MATRIX MAPS OF BOUNDED SEQUENCES IN A BANACH SPACE

I. J. MADDOX

ABSTRACT. Matrices of bounded linear operators are applied to bounded sequences in a Banach space. The main result is related to Knopp's core theorem for scalar sequences and matrices. From the main result, necessary and sufficient conditions are given for a matrix of operators to map bounded sequences into null sequences in a Banach space.

1. **Introduction.** For a real sequence $x = (x_k)$ we write $l(x) = \liminf x_k$ and $L(x) = \limsup x_k$, and for an infinite matrix $A = (a_{nk})$ of real or complex numbers, write $Ax = (\sum a_{nk}x_k)$. Now if A is regular (i.e. a Toeplitz matrix) and also nonnegative then it was shown by Knopp that

$$l(x) \leqslant l(Ax) \leqslant L(Ax) \leqslant L(x).$$

A proof may be found in [1, p. 138].

For bounded complex sequences and complex regular infinite matrices A it is shown in [1, p. 149] that the core of Ax is contained in the core of x, for all bounded sequences x, if and only if $\sum|a_{nk}| \to 1$ $(n \to \infty)$.

In this note we shall be concerned with estimates involving $p(Ax)$ and $p(x)$, where $x = (x_k)$ is a bounded sequence in a Banach space,

$$p(x) = \limsup\|x_k\|, \quad \text{and} \quad Ax = \left(\sum A_{nk}x_k\right),$$

where A_{nk} are bounded linear operators on one Banach space into another. The main result (Theorem 1 below) was motivated by the proofs of Knopp's core theorem mentioned above. However the referee has drawn my attention to a paper of Lev [2] which contains a result for scalar sequences and matrices of which my Theorem 1 is a direct generalization. My thanks are due to him for this observation and for some useful comments on notation.

2. **Notation.** Let X, Y be Banach spaces with norms $\|x\|$, $\|y\|$ which will not be distinguished, and let $B(X, Y)$ be the Banach space of bounded linear operators on X into Y, with the usual operator norm. By S we denote the set of all $x \in X$ such that $\|x\| \leqslant 1$, and by U the set of all $x \in X$ such that $\|x\| = 1$. The zero of X is denoted by θ.

Received by the editors June 29, 1976.

AMS (MOS) subject classifications (1970). Primary 40C05; Secondary 40J05.

Key words and phrases. Infinite matrices of bounded linear operators, bounded sequences in a Banach space, core theorem.

The space of bounded X-valued sequences is $l_\infty(X)$, with $\|x\| = \sup_n\|x_n\|$, where $x = (x_n) \in l_\infty(X)$. By $c_0(Y)$ we denote the null Y-valued sequences, i.e. those (y_n) such that $\|y_n\| \to 0$ $(n \to \infty)$. Define the seminorm p on $l_\infty(X)$ by $p(x) = \lim\sup\|x_n\|$.

If q is a functional on $l_\infty(X)$ and $M \geqslant 0$ is a real number, then $p \leqslant Mq$ means that $p(x) \leqslant Mq(x)$ for all $x \in l_\infty(X)$.

$A = (A_{nk})$ is an infinite matrix of elements $A_{nk} \in B(X, Y)$; the zero of $B(X, Y)$ is 0, and a statement such as $A_{nk} \to 0$ $(n \to \infty$, each $k)$ refers to the topology of pointwise convergence. We write pA for p_0A.

If $(B_k) = (B_1, B_2, \ldots)$ is a sequence in $B(X, Y)$ its group norm (see [3]) is defined to be

$$\|(B_k)\| = \sup\left\|\sum_{k=1}^{n} B_k x_k\right\|$$

where the supremum is over $n \geqslant 1$ and $x_k \in S$. Finally, write $R_{nm} = (A_{nm}, A_{n,m+1}, \ldots)$ for the mth tail of the nth row of A.

3. **A lemma.** The next result will be needed in the proof of the main theorem. We assume in the lemma that (B_k) is a sequence in $B(X, Y)$, with group norm $\|(B_k)\|$, and we write $T_m = \|(B_m, B_{m+1}, \ldots)\|$.

LEMMA 1. *$\Sigma B_k x_k$ converges, whenever $x = (x_k) \in l_\infty(X)$, if and only if* (i) $T_1 < \infty$, *and* (ii) $T_m \to 0$ $(m \to \infty)$.

PROOF. For the sufficiency, (i) implies that $T_m < \infty$ for each $m > 1$, and if $n > m$, $x \in l_\infty(X)$, then

$$\left\|\sum_{m}^{n} B_k x_k\right\| \leqslant \|x\|\cdot T_m$$

whence $\Sigma B_k x_k$ converges.

Conversely, the convergence of $\Sigma B_k x_k$ on $l_\infty(X)$ implies (i) by [4, Theorem III], or by the Banach-Steinhaus theorem. Hence $T_m < \infty$ for each $m > 1$. Suppose, if possible, that $\lim\sup_m T_m = 3c > 0$. Then there exist $n(1) \geqslant m(1) > 1$ and $x_{m(1)}, \ldots, x_{n(1)} \in S$ such that

$$\left\|\sum_{m(1)}^{n(1)} B_k x_k\right\| > c.$$

Choose $m(2) > n(1)$ such that $T_{m(2)} > 2c$. Then there exist $n(2) \geqslant m(2)$ and $x_{m(2)}, \ldots, x_{n(2)} \in S$ such that

$$\left\|\sum_{m(2)}^{n(2)} B_k x_k\right\| > c.$$

Proceed in this way, and define $z_k = \theta$ $(k < m(1))$, $z_k = x_k$ $(m(1) \leqslant k \leqslant n(1))$, $z_k = \theta$ $(n(1) < k \leqslant m(2))$, etc. Then $\|z\| \leqslant 1$, but $\Sigma B_k z_k$ diverges, which proves the lemma.

We remark that (i) of the lemma is not by itself sufficient, in general spaces X, for the convergence of $\sum B_k x_k$. For example, take X to be the space $l_\infty(C)$ of bounded sequences of complex numbers, and define $B_k z = z_1 e_k$ where $z = (z_k) \in l_\infty(C)$ and e_k is the kth unit vector in $l_\infty(C)$. Then (i) holds but $\sum B_k e_1 = B_1 e_1 + B_2 e_1 + \dots$ diverges.

4. The main theorem.

THEOREM 1. *Let $M \geqslant 0$. Then $pA \leqslant Mp$ if and only if*

(i) $A_{nk} \to 0$ $(n \to \infty$, *each* $k)$,

(ii) $\|R_{n1}\| < \infty$ *and* $\|R_{nm}\| \to 0$ $(m \to \infty$, *each* $n)$,

(iii) $\lim_m \limsup_n \|R_{nm}\| \leqslant M$.

PROOF. Let (i) to (iii) hold, and take $x \in l_\infty(X)$. By Lemma 1, $y_n = \sum A_{nk} x_k$ converges for each n, and

$$\|y_n\| \leqslant \sum_{k<m} \|A_{nk} x_k\| + \sup_{k \geqslant m} \|x_k\| \cdot \|R_{nm}\|.$$

Applying the operator $\lim_m \limsup_n$ to both sides of this inequality, (i) and (iii) imply that $p(Ax) \leqslant Mp(x)$, so that $pA \leqslant Mp$.

Conversely, let $pA \leqslant Mp$. Take $z \in X$ and define $x_k = z$, $x_n = \theta$ $(n \neq k)$. Then $p(x) = 0$ and so (i) holds. The assumption $pA \leqslant Mp$ implies the convergence for each n of $\sum A_{nk} x_k$ on $l_\infty(X)$, whence (ii) follows from Lemma 1.

Also, $\sum A_{nk} x_k = O(1)$ on $l_\infty(X)$, so that $\sup_n \|R_{n1}\| < \infty$, by the Banach-Steinhaus theorem. Since $\|R_{n,m+1}\| \leqslant \|R_{nm}\|$, the limit in (iii) exists. Denote this limit by H. If $H = 0$ then (iii) is obviously true. Suppose then that $H > 0$ and write

$$a(m) = \limsup_n \|R_{nm}\|.$$

Then $a(1) \geqslant H > 0$, and so there exist $n(1)$, $k(1)$, and $z_1, \dots, z_{k(1)} \in S$ such that

$$\left\| \sum_{k=1}^{k(1)} A_{n(1)k} z_k \right\| > H - H/2. \tag{1}$$

Not all the z_k are θ, so there exist $x_k \in S$ with at least one $x_k \in U$ such that (1) holds with x_k in place of z_k.

Now $a(k(1) + 1) \geqslant H$, so there exists $n(2) > n(1)$ such that $\|R_{n(2),k(1)+1}\| > H - H/2^3$ and

$$\left\| \sum_{k=1}^{k(1)} A_{n(2)k} x_k \right\| < H/2^2.$$

Hence there exists $q > k(1)$ and $x_{k(1)+1}, \dots, x_q \in S$, with at least one $x_k \in U$ such that

$$\left\| \sum_{k(1)+1}^{q} A_{n(2)k} x_k \right\| > H - H/2^2.$$

Since (ii) holds we may choose $k(2) > q$ such that

$$\|R_{n(2),k(2)+1}\| < H/2^2.$$

Define $x_k = \theta$ $(q < k \leqslant k(2))$, so that

$$\left\| \sum_{k(1)+1}^{k(2)} A_{n(2)k} x_k \right\| > H - H/2^2.$$

Proceed in this way and obtain $x = (x_k)$, with $\|x_k\| \leqslant 1$ for all k, and $\|x_k\| = 1$ for infinitely many k. Then $p(x) = 1$, and $n(i) < n(i+1)$, $k(i) < k(i+1)$, so that

$$\left\| \sum_{k=1}^{\infty} A_{n(i)k} x_k \right\| \geqslant \left\| \sum_{k(i-1)+1}^{k(i)} A_{n(i)k} x_k \right\| - \left\| \sum_{1}^{k(i-1)} A_{n(i)k} x_k \right\| - \|R_{n(i),k(i)+1}\|$$
$$> H - H/2^i - h/2^i - H/2^i.$$

Consequently $p(Ax) \geqslant H$, and so $M \geqslant H$, which proves the theorem.

Taking $M = 0$ in Theorem 1 we immediately obtain the

COROLLARY. *An infinite matrix $A = (A_{nk})$ of elements of $B(X, Y)$ maps $l_\infty(X)$ into $c_0(Y)$ if and only if*

$$A_{nk} \to 0 \qquad (n \to \infty, \text{ each } k),$$
$$\|R_{n1}\| < \infty \quad \text{and} \quad \|R_{nm}\| \to 0 \qquad (m \to \infty, \text{ each } n),$$
$$\lim_m \limsup_n \|R_{nm}\| = 0.$$

In the case when $X = Y = C$, the Banach space of complex numbers and the A_{nk} may be identified with complex numbers a_{nk}, the conditions in the corollary reduce to

$$(2) \qquad \sum |a_{nk}| \to 0 \qquad (n \to \infty),$$

on using the fact that $\|R_{nm}\| = \sum_{k=m}^{\infty} |a_{nk}|$. Of course (2) is the well-known necessary and sufficient condition for $A\colon l_\infty \to c_0$.

It is clear that the two conditions:

$$(3) \qquad \|R_{nm}\| \to 0 \qquad (m \to \infty, \text{each } n) \quad \text{and} \quad \|R_{n1}\| \to 0 \qquad (n \to \infty)$$

are sufficient for $A = (A_{nk})$ to map $l_\infty(X)$ into $c_0(Y)$. However, the following example shows that, in general, (3) is not necessary. Let $X = Y = c_0(C)$ and define

$$A_{nk} z = (z_n/k) e_{n+k}$$

for each $z = (z_k) \in X$. Then $\|R_{nm}\| = 1/m$, so the first condition of (3) holds but the second does not.

REFERENCES

1. R. G. Cooke, *Infinite matrices and sequence spaces*, Macmillan, London, 1950. MR **12**, 694.

2. J. Lev, *Effects of linear transformations on the divergence of bounded sequences and functions*, Trans. Amer. Math. Soc. **35** (1933), 888–896.

3. G. G. Lorentz and M. S. Macphail, *Unbounded operators and a theorem of A. Robinson*, Trans. Roy. Soc. Canada Sec. III (3) **46** (1952), 33–37. MR **14**, 634.

4. A. Robinson, *On functional transformations and summability*, Proc. London Math. Soc. (2) **52** (1950), 132–160. MR **12**, 253.

DEPARTMENT OF PURE MATHEMATICS, QUEEN'S UNIVERSITY OF BELFAST, BELFAST BT7 1NN, NORTHERN IRELAND