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ON THE HEIGHTS OF GROUP CHARACTERS

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ABSTRACT. For a finite p-soluble group G we derive a bound on the heights of the irreducible complex characters of G lying in a p-block B. This bound depends on the prime p and the exponent d of a defect group of B. We show by examples that this bound is of the right order of magnitude.

Let G be a finite group of order $p^e g_0$, where p is a fixed prime, e is an integer $\geqslant 0$, and $(g_0, p) = 1$. In the theory of modular representations, the characters of the irreducible complex representations of G may be partitioned into disjoint sets, the so-called *blocks* of G for the prime p. Associated with each block B is a p-subgroup D of G of order p^d, unique up to conjugacy in G. d is called the *defect* of the block B. If χ is an irreducible complex character of G, or as we shall say, an ordinary character of G, and χ lies in a block B, written $\chi \in B$, then χ has degree divisible by p to the exponent $(e - d + h(\chi))$. The nonnegative integer $h(\chi)$ is called the *height* of χ.

In [4] Fong proves the following: Let G be a finite p-soluble group and B be a block of G for the prime p. Suppose that B has defect group D and let $Z(D)$ denote the centre of D. Then for each ordinary character $\chi \in B$ we have $h(\chi) \leqslant \nu_p(|D : Z(D)|)$ where $\nu_p(t)$ denotes the exponent of p dividing t.

A slight modification of Fong's proof yields that for $d \geqslant 2$, $h(\chi)$ never exceeds $(d - 2)$. Brauer and Feit [2] obtain this bound for an arbitrary finite group. In this paper we prove the following result.

THEOREM. *Let G be a finite p-soluble group with a block B of defect $d \geqslant 2$. Then there exists a function $f(p, d)$ such that $h(\chi) \leqslant f(p, d)$ for all ordinary characters $\chi \in B$;*

$$f(p,d) = \begin{cases} \left(\dfrac{3d-4}{4}\right) & \textit{if } p = 2, \\[2ex] \dfrac{(p^2+1)}{(p^2-p+1)}\left(\dfrac{d-1}{2}\right) & \textit{if } p \textit{ is an odd Fermat prime}, \\[2ex] \dfrac{(p+1)}{(p)}\,\dfrac{(d-1)}{2} & \textit{if } p \textit{ is any other prime}. \end{cases}$$

We give examples to show that this bound is of the right order of magnitude.

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LEMMA 1. *Let G be a finite p-soluble group and B be a block of G with defect group D, of order p^d. Suppose that $H \lhd G$, then there exists an irreducible constituent of $\chi|_H$, say θ, such $h(\chi) \leq h(\theta) + \nu_p(|G:H|)$. $h(\theta)$ denotes the height of θ over the block b of H with $\theta \in b$ and $h(\chi)$ denotes the height of χ over B.*

PROOF. Choose a series $G = G_1 \geq G_2 \geq \cdots \geq G_r = H$ such that G_{i+1} is a maximal normal subgroup of G_i for $i = 1, \ldots, r-1$. Define $e_i = \nu_p(|G_i|)$. Now choose ordinary characters $\chi_1, \ldots, \chi_r$ such that χ_i is an ordinary character of G_i and $\chi = \chi_1$, and χ_{i+1} is an irreducible constituent of χ_i restricted to G_{i+1}. Now there exist blocks $B_1, \ldots, B_r$ with B_i a block of G_i containing χ_i for $i = 1, \ldots, r$. Let $\theta = \chi_r$ and hence $B_r = b$. These conditions mean that B_i covers B_{i+1} in the sense of Brauer [1]. Finally let D_i be a defect group of B_i for each i and suppose that d_i = defect of B_i. We have

$$\nu_p(\deg \chi_i) = e_i - d_i + h(\chi_i) \qquad \text{for } i = 1, \ldots, r.$$

When $|G_i : G_{i+1}|$ is coprime to p then by Cliffords theorem $\nu_p(\deg \chi_i)$ equals $\nu_p(\deg \chi_{i+1})$. Clearly $e_i = e_{i+1}$ and by [1, 2E] $d_i = d_{i+1}$. Thus $h(\chi_i) = h(\chi_{i+1})$. Otherwise $|G_i : G_{i+1}| = p$ and Cliffords theorem yields that

$$\nu_p(\deg \chi_i) \leq \nu_p(\deg \chi_{i+1}) + 1.$$

Also $e_i = e_{i+1} + 1$ and by [3], $d_i \leq d_{i+1} + 1$. We conclude that in this case $h(\chi_i) \leq h(\chi_{i+1}) + 1$.

Hence $h(\chi) = h(\chi_1) \leq h(\chi_r) + \nu_p(|G:H|) = h(\theta) + \nu_p(|G:H|)$ as required.

LEMMA 2. *If G is a finite p-soluble group which is faithfully and irreducibly represented on a vector space V of dimension n over $GF(p)$ then $\nu_p(|G|)$ does not exceed $\lambda(p, n)$ where*

$$\lambda(p,n) = \begin{cases} (n-1) & \text{if } p = 2, \\ (np)/(p-1)^2 & \text{if } p \text{ is an odd Fermat prime}, \\ n/(p-1) & \text{otherwise}. \end{cases}$$

PROOF. $\nu_2(|G|) \leq n - 1$ by Huppert [7, Satz 14]. For odd p a paper of Winter [8] yields that

$$\nu_p(|G|) \leq \begin{cases} \displaystyle\sum_{i=0}^{\infty} \frac{n}{p^i(p-1)} & \text{for } p \text{ Fermat}, \\ \displaystyle\sum_{i=1}^{\infty} \frac{n}{p^i} & \text{for } p \text{ not Fermat}. \end{cases}$$

Since for $|x| < 1$, $\sum_{i=0}^{\infty} x^i = 1/(1-x)$ our lemma follows easily.

LEMMA 3. *If G is a finite p-soluble group with $O_{p'p}(G) \cong O_{p'}(G) \times O_p(G)$ and if $|O_p(G) : \Phi(O_p(G))| = p^n$ where $|O_p(G)| = p^m$ then*

$$\nu_p(|G|) \leq m + \lambda(p, n).$$

PROOF. By [5, 1.2.5] $G/O_{p'p}(G)$ is faithfully represented on $O_{p'p}(G)/F$ where $F/O_{p'}(G) = \Phi(O_{p'p}(G)/O_{p'}(G))$. Thus under our hypotheses $G/O_{p'p}(G)$ is faithfully represented on $O_p(G)/\Phi(O_p(G))$. Let $L_1, \dots, L_s$ denote the p-chief factors of G lying between $O_p(G)$ and $\Phi(O_p(G))$. Then $C = \bigcap_{i=1}^{s} C_G(L_i) \geqslant O_{p'p}(G)$ since $O_{p'p}(G)$ centralizes all p-chief factors. For each $i = 1, \dots, s$ we have that $G/C_G(L_i)$ is a faithful irreducible subgroup of $GL(n_i, p)$ where n_i is just the rank of L_i. Since $C/O_{p'p}(G)$ is merely a group of automorphisms of a p-group which stabilizes a normal series for that group we conclude that $C/O_{p'p}(G)$ is a p-group and thus $C = O_{p'p}(G)$.

Now $G/O_{p'p}(G)$ is isomorphic to a subgroup of $G/C_G(L_1) \times \cdots \times G/C_G(L_s)$ so in particular

$$\nu_p(|G : O_{p'p}(G)|) \leqslant \sum_{1}^{s} \nu_p(|G : C_G(L_i)|).$$

Since $\lambda(p, k)$ is linear in the second variable, and using Lemma 2 we see that $\nu_p(|G|) \leqslant m + \lambda(p, n)$ as required.

PROOF OF THEOREM. We proceed by induction on the order of G. By [4, 2B and 2D] we may assume that all blocks of G have maximal defect, so $d = \nu_p(|G|)$. Furthermore $O_{p'}(G)$ is cyclic and central in G so $O_{p'p}(G) = O_{p'}(G) \times O_p(G)$. Let $H = O_p(G)$ and set $|H| = p^m$ and $|H : \Phi(H)| = p^n$. Let θ be an irreducible constituent of $\chi|_H$. Now $\nu_p(|G : H|) = d - m$ and thus by Lemma 1 $h(\chi) \leqslant h(\theta) + d - m$. We consider two possibilities:

(a) *H is abelian*. In this case $h(\theta) = 0$ and by Lemma 3, since $n \leqslant m$ we have that $d \leqslant m + \lambda(p, m)$. When $p = 2$, $\lambda(2, m) = m - 1$ and so $d \leqslant 2m - 1$. Hence $h(\chi) \leqslant (d - 1)/2$. For p an odd Fermat prime a similar calculation yields $h(\chi) \leqslant (pd)/(p^2 - p + 1)$. Finally for p odd and not Fermat, using Lemma 2 again we deduce that $h(\chi) \leqslant d/p$. These three bounds are less than the ones appearing in the statement of the theorem.

(b) *H is nonabelian*. Now $h(\theta) \leqslant (m - 1)/2$ since θ is a character of a nonabelian p-group of order p^m. Also H nonabelian implies that $n \leqslant m - 1$. Thus $h(\chi) \leqslant d - (m + 1)/2$ by Lemma 1 and $d \leqslant m + \lambda(p, m - 1)$ by Lemma 3. For $p = 2$, $d \leqslant 2(m - 1)$ and so $h(\chi) \leqslant (3d - 4)/4$. When p is an odd Fermat prime then $d \leqslant (m(p^2 - p + 1) - p)/(p - 1)^2$ and thus $h(\chi) \leqslant ((p^2 + 1)/(p^2 - p + 1))(d - 1)/2)$. Finally for non-Fermat primes p, $d \leqslant (mp - 1)/(p - 1)$ and a brief calculation yields

$$h(\chi) \leqslant ((p + 1)/p)((d - 1)/2).$$

Our theorem is now proved.

The theorem is best possible in the following sense: Given an odd integer $d \geqslant 1$ choose p to be a prime with $p > d$. Now there exists an extraspecial p-group of order p^d; this group possesses an ordinary character of height $\frac{1}{2}(d - 1)$. We have so chosen things that $\frac{1}{2}(d - 1)$ is the greatest integer less than $f(p, d)$. We give less trivial examples for $p = 2$ and $p = 3$ below.

(1) Let $G_i \cong \mathrm{GL}(2, 3)$, the group of 2×2 matrices over the Galois field of three elements, for $i = 1, 2, \dots, n$. Form the central product $G(n) = G_1 \curlyvee G_2 \curlyvee \cdots \curlyvee G_n$ (see [6, I.9.10]). Let $Z(G_i) = gp\{z_i : z_i^2 = 1\}$ then $G(n) \cong (G_1 \times G_2 \times \cdots \times G_n)/\Delta$ where $\Delta = gp\{z_1 z_2^{-1}, z_2 z_3^{-1}, \dots, z_n z_1^{-1}\}$. Now G

is a 2-soluble group and $O_{2'}(G(n)) = 1$. Thus $G(n)$ possesses one block for the prime 2 and hence this has defect $d = 3n + 1$. G_i has a character θ_i of degree four such that $\theta_i(z_i) = -4$. Form $\theta_1 \otimes \theta_2 \otimes \cdots \otimes \theta_n = \chi$, an irreducible character of $G(n)$, since one easily checks that $\Delta \leqslant \operatorname{Ker}\chi$. We have that $h(\chi) = 2n = 2(d-1)/3$.

(2) Let E be the extraspecial 3-group of order 27 and exponent 3. Since $\mathrm{SL}(2,3)$ is isomorphic to a subgroup of the automorphism group of E, we may form the semidirect product of E by $\mathrm{SL}(2,3)$. Denote this group by H. H has a centre of order 3 and furthermore has an irreducible character of degree 9, say α. If $Z(H) = gp\{z_i : z_i^3 = 1\}$ then $\alpha(z_i) = 9\omega$ and $\alpha(z_i^2) = 9\omega^2$ where ω is a primitive cube root of unity. As in the previous example we construct $H(n)$, the central product of n copies of H. $H(n)$ is a 3-soluble group in which $O_{3'}(H(n)) = 1$ and so $H(n)$ has a unique block for the prime 3 and this has defect $d = 3n + 1$. The character $\psi =$ tensor product of n copies of α, is an irreducible character of $H(n)$. ψ has height $2n = 2(d-1)/3$.

We have shown that if $g(p,d)$ is the precise bound on the heights of characters of p-soluble groups then

$$2(d-1)/3 \leqslant g(2,d) \leqslant 3(d-1)/4,$$
$$2(d-1)/3 \leqslant g(3,d) \leqslant 5(d-1)/7.$$

In fact there exist examples for all primes p of p-soluble groups G with a p-block B, of defect d, containing an ordinary character of height exceeding $(d-1)/2$.

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