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SHORTER NOTES

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A SIMPLE PROOF OF A THEOREM OF KY FAN

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ABSTRACT. A simple proof of a fixed point theorem of Ky Fan in locally convex spaces is constructed by means of his fixed point theorem for set valued functions.

Let X and Y be topological spaces. A set valued function f: $X \to 2^Y$ is upper semicontinuous, u.s.c. (resp. lower semicontinuous, l.s.c.) iff for each closed (open) subset B of Y, the set $f^{-1}(B) = \{x: f(x) \cap B \neq \emptyset\}$ is a closed (open) subset of X. We shall give a simple proof of the following theorem of Ky Fan [3] using one of his well-known fixed point theorems for set valued functions [2].

THEOREM. *Let X be a nonempty compact and convex subset of a locally convex, Hausdorff topological vector space E and let f: $X \to E$ be a continuous mapping. Then either* (a) *f has a fixed point in X or* (b) *there exist an $x \in X$ and a continuous seminorm p on E satisfying*

$$0 < p(x - f(x)) = \min\{p(y - f(x)): y \in X\}.$$

PROOF. Let $\mathcal{P}$ denote the family of all continuous seminorms on E. Let $p \in \mathcal{P}$. Define a mapping $m = m_p$: $X \to R^+$ (nonnegative reals) by

$$m(x) = \min\{p(y - f(x)): y \in X\}. \tag{1}$$

It follows (see Example 9, p. 252 in [1]) that m is continuous and for each $x \in X$ there is a $y(x) \in X$ such that

$$p(y(x) - f(x)) = m(x). \tag{2}$$

Define a set function $g = g_p$: $X \to 2^X$ by

$$g(x) = \{y \in X: p(y - f(x)) = m(x)\}. \tag{3}$$

Then by (2), $g(x) \neq \emptyset$ and it is clear by (3) that $g(x)$ is a closed and convex subset of X for each $x \in X$. We show that g is u.s.c. Let A be a closed subset

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of X and let a net $\{x_\alpha: \alpha \in \Gamma\} \subseteq g^{-1}(A)$ converge to an $x_0 \in X$. This implies that for each $\alpha \in \Gamma$, there is a $y_\alpha \in g(x_\alpha) \cap A$, that is, $y_\alpha \in A$ and

$$p(y_\alpha - f(x_\alpha)) = m(x_\alpha).$$

Now, A being compact, there exist a $y_0 \in A$ and a subnet $\{y_\delta\}$ of the net $\{y_\alpha\}$ such that $y_\delta \to y_0$ and, hence, $p(y_0 - f(x_0)) = m(x_0)$, that is, $y_0 \in g(x_0) \cap A$. This implies that $x_0 \in g^{-1}(A)$. Thus g is u.s.c. Therefore, by Ky Fan's Theorem 1 in [2], for each $p \in \mathcal{P}$, there exists an $x_p \in X$ such that $x_p \in g_p(x_p)$, that is,

$$p(x_p - f(x_p)) = m(x_p). \tag{4}$$

Now, if $m(x_p) > 0$ for some $p \in \mathcal{P}$, then (4) implies (b) for this p. If $m(x_p) = 0$ for each $p \in \mathcal{P}$, then by (4) $F_p = \{x \in X: p(x - fx) = 0\}$ is a nonempty compact subset of X for each $p \in \mathcal{P}$. Further, since for any finite family $\Delta \subseteq \mathcal{P}$, $p_0 = \sum_{p \in \Delta} p \in \mathcal{P}$, it follows that the family $F = \{F_p: p \in P\}$ has a finite intersection property. Consequently, there is an $x \in X$ such that $p(x - f(x)) = 0$ for each $p \in \mathcal{P}$. Thus f satisfies (a) for this $x \in X$.

It may be remarked that the arguments in the above proof remain valid if f therein is replaced by a continuous (u.s.c. and l.s.c.) set valued function $f: X \to 2^E$ with $f(x)$ a compact and convex subset of E for each $x \in X$, where in this case $p(y - f(x)) = \min\{p(y - z): z \in f(x)\}$. Thus, the above theorem can be extended to such set valued functions.

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