PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 63, Number 2, April 1977

THE NORMED SPACE NUMERICAL INDEX OF C^*-ALGEBRAS

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ABSTRACT. Given a complex C^*-algebra X, we prove that the normed space numerical index $n(X)$ of X is 1 or $\frac{1}{2}$ according as X is commutative or not commutative.

Let X be a normed linear space, X^* its dual space, and $B(X)$ the normed algebra of all bounded linear operators on X with operator norm. Given $T \in B(X)$, the numerical range $V(T)$, and the numerical radius $v(T)$ of T are defined by

$$V(T) = \{f(Tx): x \in X, f \in X^*, f(x) = \|x\| = \|f\| = 1\},$$
$$v(T) = \sup\{|\lambda|: \lambda \in V(T)\}.$$

The numerical index $n(X)$ of the space X is defined by

$$n(X) = \inf\{v(T): T \in B(X), \|T\| = 1\}.$$

Standard results for $n(X)$ are given in [1, §32].

THEOREM. *Let X be a complex C^*-algebra. Then $n(X)$ is 1 or $\frac{1}{2}$ according as X is commutative or not commutative.*

PROOF. If X is commutative, $n(X) = 1$ by [1, Theorem 32.8]. If X is not commutative, the numerical index of X *as an algebra* is $\frac{1}{2}$ by [2, Theorem 3]. Since the left regular representation of X is isometric, it follows that $n(X) \leqslant \frac{1}{2}$. Let $T \in B(X)$ with $\|T\| = 1$. It remains to show that $v(T) \geqslant \frac{1}{2}$. We may assume (by embedding X in its Arens second dual) that X is unital. Let $\varepsilon > 0$. Choose $y \in X$ with $\|y\| = 1$, $\|Ty\| > 1 - \varepsilon$. By [3, Theorem 1] there exist positive real numbers $\alpha_1, \ldots, \alpha_n$ with $\sum_{j=1}^n \alpha_j = 1$ and unitary elements $u_1, \ldots, u_n$ of X such that $\|y - \sum_{j=1}^n \alpha_j u_j\| < \varepsilon$. For some j with $1 \leqslant j \leqslant n$ we have $\|Tu_j\| > 1 - 2\varepsilon$. Choose a state g of X such that $|g((Tu_j)u_j^*)| > \frac{1}{2}(1 - 2\varepsilon)$. Then $(u_j^* g)(u_j) = g(u_j u_j^*) = 1$, and so $(u_j^* g)(Tu_j) \in V(T)$. This gives $v(T) \geqslant \frac{1}{2}(1 - 2\varepsilon)$. The proof is complete.

REMARK. The final step in the proof is based on an idea of Crabb (see [1, Theorem 32.9]).

Received by the editors July 30, 1976 and, in revised form, November 22, 1976.
AMS (MOS) subject classifications (1970). Primary 46L05, 47A10.
Key words and phrases. C^*-algebra, numerical index, numerical radius, numerical range.

The author wishes to thank the referee for giving a simplified version of this paper.

REFERENCE

1. F. F. Bonsall and J. Duncan, *Numerical ranges*. II, London Math. Soc. Lecture Note Series, 10, Cambridge Univ. Press, London, 1973.

2. M. J. Crabb, J. Duncan and C. M. McGregor, *Characterizations of commutativity for* C^*-*algebras*, Glasgow Math. J. **15** (1974), 172–175. MR **50** #14252.

3. B. Russo and H. A. Dye, *A note on unitary operators in* C^*-*algebras*, Duke Math. J. **33** (1966), 413–416. MR **33** #1750.

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