PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 63, Number 2, April 1977

COMPACTIFICATION BY THE TOPOLOGIST'S SINE CURVE

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ABSTRACT. Using a compactification of the nonnegative reals whose remainder is the topologist's sine curve, results about growths of Stone-Čech compactifications are proved. For example, it is proved that if βX contains a nonconstant continuous image of a compact connected LOTS, then the image is contained in υX. This extends a result of Peter Nyikos.

In this note we discuss some of the consequences of the fact that the topologist's sine curve is $CR^+ - R^+$ for a particular compactification CR^+ of the nonnegative real numbers. In particular, we give a new proof of the fact that if $\beta X - X$ is path-connected, then X is pseudocompact. We also give an apparently new proof of the fact, which was communicated in a letter by Peter Nyikos, that if βX contains a nonconstant path then the path is contained in υX.

The author wishes to thank Peter Nyikos and the referee for many helpful suggestions which generalized the original versions of the theorems. In particular, in their original forms the results were stated for T-paths rather than locally connected continua.

1. **Preliminaries.** For general definitions, see [4]. All spaces are completely regular Hausdorff. If X is any space, βX denotes the Stone-Čech compactification of X and υX denotes the Hewitt realcompactification of X. A continuum is a compact connected space. R denotes the set of real numbers and R^+ denotes the set of nonnegative real numbers. The topologist's sine curve is the space $S = S_1 \cup S_2$ where

$$S_1 = \{0\} \times [-1, 1] \quad \text{and} \quad S_2 = \{(x, \sin(1/x)) \colon 0 < x \leqslant 1\}.$$

S is given the induced topology from R^2.

It is known that S is the remainder of some compactification of R^+ (see, for example, [4]). It is convenient for the sake of reference, however, to describe a particular compactification, which we call CR^+, such that $CR^+ - R^+$ is S.

We observe that the minima of the function $\sin(1/x)$ (defined on (0, 1]) are at $x = 2/[(4n-1)\pi]$ for $n = 1, 2, \ldots$, and the minimum value is -1. For $n = 1, 2, \ldots$, let $B_n = \{(x, 1/n) \colon 2/[(4n-1)\pi] \leqslant x \leqslant 1\}$, so for each n, B_n

Received by the editors October 13, 1975 and, in revised form, June 1, 1976.
AMS (MOS) subject classifications (1970). Primary 54D40, 54F15.
Key words and phrases. T-path, Stone-Čech compactification.

is a horizontal line segment which is $1/n$ units above the x-axis. Let C_n be the line segment joining the left endpoint of B_n to the right endpoint of B_{n+1}. Formally,

$$C_n = \left\{ \left(x, \frac{\pi(1-4n)(x-1)}{n(n+1)[(4n-1)\pi - 2]} + \frac{1}{n+1} \right) : \frac{2}{(4n-1)\pi} \leqslant x \leqslant 1 \right\}.$$

Let $A = \bigcap_{n=1}^{\infty}(B_n \cup C_n)$. To obtain CR^+ we "bend" A in such a way that it conforms to S. Let $L = \{(x, \sin(1/x)) + (x, y): (x, y) \in A\}$. L is clearly homeomorphic to R^+. Let $CR^+ = S \cup L$. Then CR^+ is clearly a compactification of R^+ (where we view R^+ as L) and $CR^+ - L = S$. For $k = 1, 2, \ldots$ let $p_k = (2/[(4k-1)\pi], (1/k) - 1)$; then the p_k's are the "sharp" points of L (that is, the nondifferentiable points of L) whose x-coordinates are not 1.

2. **Path-connectedness properties of βR^+.** For the remainder of this paper let $T = [a, b]$ be any compact connected linearly ordered topological space with $a < b$.

LEMMA 2.1. *If K is a locally connected continuum and f: $K \to S$ is continuous, then either $f(K) \subseteq S_1$ or $f(K) \subseteq S_2$.*

PROOF. Any locally connected continuum contained in S is clearly contained in either S_1 or S_2, and local connectedness is preserved under closed continuous maps (see [2] for example).

THEOREM 2.2. *If K is a nontrivial continiuum contained in $\beta R^+ - R^+$, then there is a continuous function G: $K \to S$ whose range meets both S_1 and S_2.*

PROOF. Suppose $p, q \in K, p \neq q$. Let $\tilde{\tilde{U}}$ and $\tilde{\tilde{V}}$ be disjoint closed βR^+-neighborhoods of p and q and let $U = \tilde{\tilde{U}} \cap R^+$, $V = \tilde{\tilde{V}} \cap R^+$. Choose an increasing cofinal sequence $a_1 < a_2 < \ldots$ in R^+ such that

$$a_i \in R^+ \setminus (U \cup V) \quad \text{for each } i.$$

Define g: $U \cup V \to L$ by $g|[a_n, a_{n+1}] \cap U = (1, (\sin 1) + 1)$, $g|[a_n, a_{n+1}] \cap V = p_n$. Since the sets $[a_n, a_{n+1}] \cap U$ and $[a_n, a_{n+1}] \cap V$ are disjoint and open in $U \cup V$, g is well defined and continuous. g extends continuously to $\tilde{g}$: $R^+ \to L$. $\tilde{g}$ extends continuously to $\tilde{g}^*$: $\beta R^+ \to CR^+$. Then $G = \tilde{g}^*|K$ is the required function.

DEFINITION. If X is a space and f: $T \to X$ is continuous, f is called a *T-path* joining $f(a)$ to $f(b)$; its image, which is also called a T-path, is denoted $\tilde{T}$.

Combining Lemma 2.1 and Theorem 2.2 for the special case when K is a T-path gives the following:

COROLLARY 2.3. (*See also*[1].) *$\beta R^+ - R^+$ contains no nonconstant T-path.*

PROPOSITION 2.4. *If βR^+ contains a nontrivial locally connected continuum K, $K \subseteq R^+$.*

Proof. It suffices to show that there can be no pair $a, b \in K$ with $a \in R^+, b \in \beta R^+ - R^+$. Suppose there were such a pair; let g be a continuous map from βR^+ to S such that $g|R^+$ is a homeomorphism onto S_2. Then $g(b) \in S_1, g(a) \in S_2$, contradicting 2.1.

3. **T-paths in βX.** A special case of the following theorem was given in a letter by Peter Nyikos, but his proof was quite different.

Theorem 3.1. *If X is realcompact and f: $T \to \beta X$ is a nonconstant T-path, then $\tilde{\tilde{T}} \subseteq X$.*

Proof. Suppose $p \neq q, p, q \in \tilde{\tilde{T}}, p \in \beta X - X$. There is a $g \in C(\beta X)$ such that $0 \leqslant g \leqslant 1, g(q) = 0, g(p) = 1$. Since X is realcompact, there is an $h \in C(X)$, $h \geqslant 0$, such that h does not extend to p. Let $F = (g|X)(h|X)$. Then F extends to q (or is already defined at q) but F does not extend to p. We now view F as a function from X to βR^+ and let F^*: $\beta X \to \beta R^+$ be its Stone extension. Then $F^*(q) = 0, F^*(p) \notin R^+$ so F^* is a nonconstant T-path containing points of $\beta R^+ - R^+$, contradicting Corollary 2.3.

Corollary 3.2. *If $\tilde{\tilde{T}}$ is a nonconstant T-path in βX, then $\tilde{\tilde{T}} \subseteq \upsilon X$.*

Proof. $\beta(\upsilon X) = \beta X$. Apply Theorem 3.1 to υX.

Remark. We note that Theorem 3.1 and Corollary 3.2 hold with "nonconstant T-path" replaced with "nontrivial locally connected continuum."

Corollary 3.3. *If $\beta X - X$ is path connected, or if βX is path connected, or if every point of $\beta X - X$ is an element of some nonconstant T-path of βX for some T, then X is pseudocompact.*

Proof. By Corollary 3.2, if every point of $\beta X - X$ is an element of a nonconstant T-path, then every point of $\beta X - X$ is in υX, that is $\beta X = \upsilon X$.

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