PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 63, Number 2, April 1977

$\aleph_0$-CATEGORICITY OF PARTIALLY ORDERED SETS OF WIDTH 2

J. H. SCHMERL[1]

ABSTRACT. A result of J. Rosenstein is that every $\aleph_0$-categorical theory of linear order is finitely axiomatizable. We extend this to the case of partially ordered sets of width 2.

In [4] J. Rosenstein proved that every $\aleph_0$-categorical theory of linear order is finitely axiomatizable. Later in [6] we gave a proof of this fact utilizing the notion of a nuclear structure. Our method allowed us to extend Rosenstein's result to trees: we proved that every finite-branching $\aleph_0$-categorical tree has a finitely axiomatizable theory, and that every $\aleph_0$-categorical tree has a decidable theory. In this paper we again employ nuclear structures to extend Rosenstein's theorem in another way. Recall that a partially ordered set has width $\leqslant n$ if it has no antichain of length $n + 1$. Our main result, proved in §2, is that every $\aleph_0$-categorical, partially ordered set of width 2 has a finitely axiomatizable theory.

Let us recall the definition of a nuclear structure, introduced in [6]. Suppose T is a complete theory. As usual, p is an n-type if it is a maximal set of formulas consistent with T, where the free variables in each formula are from the set $\{x_0, \ldots, x_{n-1}\}$. If $I \subseteq n$, then let $p|I$ be the set of formulas in p involving only the variables in $\{x_i: i \in I\}$. Now let $\mathfrak{A}$ be a model of T, and suppose $X = \{a_0, \ldots, a_{m-1}\} \subseteq A$, $I = \{i_0, \ldots, i_{n-1}\}$, where $i_0 < \cdots < i_{n-1} < m$, $Y = \{a_i: i \in I\}$ and $a \in A$. Then we say that Y is a *nucleus of X for a* if the following holds: if p is the $(m + 1)$-type realized by $\langle a_0, \ldots, a_{m-1}, a\rangle$, then p is the unique $(m + 1)$-type extending $p|m \cup p|(I \cup \{m\})$. We say that $\mathfrak{A}$ is *n-nuclear* if for every finite $X \subseteq A$ and $a \in A$, there is a nucleus Y of X for a such that $|Y| \leqslant n$. If $\mathfrak{A}$ is n-nuclear for some $n < \omega$, then $\mathfrak{A}$ is *nuclear*. The relevant fact about nuclear structures is that if $\mathfrak{A}$ is $\aleph_0$-categorical and nuclear, and the language of $\mathfrak{A}$ is finite, then $\mathrm{Th}(\mathfrak{A})$ is finitely axiomatizable. (See [6] for details.)

The cornerstone of any investigation into $\aleph_0$-categoricity is the fundamental Ryll-Nardzewski Theorem [5]. This theorem asserts that a complete theory T

Presented to the Society, June 3, 1976; received by the editors September 13, 1976.
AMS (MOS) subject classifications (1970). Primary 02H05, 02G15; Secondary 06A10.
Key words and phrases. $\aleph_0$-categoricity, partial order, finite axiomatizability.
[1] Research partially supported by NSF Grant MCS 76-07258. The main result of this paper was announced in [7].

is $\aleph_0$-categorical iff for each $n < \omega$ the number of its n-types is finite. We will use this theorem frequently.

1. **Monotone relations on linearly ordered sets.** Let $(A, <)$ be a linearly ordered set. A binary relation $R \subseteq A \times A$ is *monotone* iff the following two conditions hold:

(1) if $(x,y) \in R$ and $x' < x$, then $(x',y) \in R$;

(2) if $(x,y) \in R$ and $y < y'$, then $(x,y') \in R$.

The purpose of this section is to prove the following theorem.

THEOREM 1. *If* $\mathfrak{A} = (A, <, R_0, \ldots, R_{n-1})$ *is* $\aleph_0$*-categorical, where* $(A, <)$ *is a linearly ordered set and each of* $R_0, \ldots, R_{n-1}$ *is monotone, then* $\mathrm{Th}(\mathfrak{A})$ *is finitely axiomatizable.*

Before beginning the proof, let us note some things about monotone relations. Consider any linearly ordered set $(B, <)$. Notice that $<$ itself is monotone, as is $\varnothing$. If R is monotone, then

$$R^* = \{(x,y) \in B \times B \colon (y,x) \notin R\}$$

is monotone. If R and S are both monotone, then their composition

$$RS = \{(x,y) \in B \times B \colon (x,z) \in R \text{ and } (z,y) \in S \text{ for some } z \in B\}$$

is also monotone. If R and S are monotone, then R^* is definable in $(B, <, R)$ and RS is definable in $(B, <, R, S)$.

We will call a structure $\mathfrak{B} = (B, <, S_0, \ldots, S_{m-1})$ a *monotone algebra* iff each of the following holds:

(1) $(B, <)$ is a linearly ordered set;

(2) each S_i is monotone;

(3) some S_i is $<$, and some $S_i = \varnothing$;

(4) for each $i < m$ there is $j < m$ such that $S_i^* = S_j$;

(5) for each $i, j < m$, there is $k < m$ such that $S_i S_j = S_k$.

Consider again the $\aleph_0$-categorical structure $\mathfrak{A}$. By Ryll-Nardzewski's Theorem there are only finitely many monotone relations definable in $\mathfrak{A}$. Thus, without loss of generality, we can assume in Theorem 1 that $\mathfrak{A}$ is in fact a monotone algebra.

For monotone R we let $R(x) = \{y \colon (x,y) \in R\}$.

LEMMA 1.1. *Suppose that* $\mathfrak{A} = (A, <, R_0, \ldots, R_{n-1})$ *is a monotone algebra, and that* $a \in A$ *and* $y \in X \subseteq A$. *Furthermore, suppose that there is an* $i < n$ *such that for any* $x \in X$ *and* $k < n$,

(i) $a \in R_k(x) \Leftrightarrow R_i(y) \subseteq R_k(x)$.

Then, for each $k, r < n$ *and* $x \in X$,

(ii) $R_r(a) \subseteq R_k(x) \Leftrightarrow R_i R_r(y) \subseteq R_k(x)$.

PROOF. Let $r < n$, and let p be such that $R_p = R_i R_r$. Since $a \in R_i(y)$ it is clear that $R_r(a) \subseteq R_p(y)$. Now suppose that $s < n$ and $w \in X$ are such that

$R_r(a) \subseteq R_s(w) \subseteq R_p(y)$. Let $R_j = (R_r R_s^*)^*$. It is easy to check that $z \in R_j(w)$ iff $R_r(z) \subseteq R_s(w)$, and hence $a \in R_j(w)$. Thus (i) implies that $R_i(y) \subseteq R_j(w)$, so that whenever $z \in R_i(y)$, then $R_r(z) \subseteq R_s(w)$. But this implies that $R_p(y) \subseteq R_s(w)$, from which it follows that $R_p(y) = R_s(w)$. □

There is a dual form of this lemma.

LEMMA 1.2. *Suppose that* $\mathfrak{A} = (A, <, R_0, \ldots, R_{n-1})$ *is a monotone algebra, and that* $a \in A$ *and* $z \in X \subseteq A$. *Furthermore, suppose that there is an* $i < n$ *such that for any* $x \in X$ *and* $k < n$,

(i) $a \notin R_k(x) \Leftrightarrow R_k(x) \subseteq R_i(z)$.

Then, for each $k, r < n$ *and* $x \in X$,

(ii) $R_k(x) \subseteq R_r(a) \Leftrightarrow R_k(x) \subseteq (R_r^* R_i^*)^*(z)$.

PROOF. Consider the "dual" monotone algebra $\mathfrak{A}' = (A, >, R'_0, \ldots, R'_{n-1})$, where $(u, v) \in R'_i$ iff $(u, v) \notin R_i$. Thus, whenever $x \in X$ and $k < n$, then $a \in R'_k(x) \Leftrightarrow R'_i(z) \subseteq R'_k(x)$. Applying Lemma 1.1, $R'_r(a) \subseteq R'_k(x) \Leftrightarrow R'_i R'_r(z) \subseteq R'_k(x)$, so that $R_k(x) \subseteq R_r(a) \Leftrightarrow R_k(x) \subseteq (R'_i R'_r)'(z)$. Finally, as is easily checked, note that $(R'_i R'_r)' = (R_r^* R_i^*)^*$. □

REMARK. The subset $R_i R_r(y)$, mentioned in Lemma 1.1, is definable in $(\mathfrak{A}, R_i(y))$. Similarly, the subset $(R_r^* R_i^*)^*(z)$, mentioned in Lemma 1.2, is definable in $(\mathfrak{A}, R_i(z))$.

Notice that whenever $a \in A$ and X is a nonempty finite subset of A, then there are $y, z \in X$ which do satisfy the hypotheses of Lemmas 1.1 and 1.2, respectively. We will refer to such a subset $\{y, z\}$ of X as a *prenucleus of X for a*.

LEMMA 1.3. *If* $\mathfrak{A} = (A, <, R_0, \ldots, R_{n-1})$ *is an* $\aleph_0$*-categorical monotone algebra, then* $\mathfrak{A}$ *is* 2*-nuclear. In fact, if* $a \in A$ *and* $X \subseteq A$ *is nonempty and finite, then any prenucleus of X for a is also a nucleus of X for a.*

PROOF. We can suppose that $\mathfrak{A}$ is countable. Consider a finite sequence $\langle x_0, \ldots, x_m \rangle$ of elements from A. Let us say that $R_p(x_i)$ and $R_q(x_j)$ are neighbors (with respect to $\langle x_0, \ldots, x_m \rangle$) if whenever $k \leqslant m$ and $r < n$ are such that either $R_p(x_i) \subseteq R_r(x_k) \subseteq R_q(x_j)$ or $R_q(x_j) \subseteq R_r(x_k) \subseteq R_p(x_i)$, then either $R_r(x_k) = R_p(x_i)$ or $R_r(x_k) = R_q(x_j)$. Let us say that $\langle x_0, \ldots, x_m \rangle$ and $\langle y_0, \ldots, y_m \rangle$ are equivalent iff whenever $R_p(x_i)$ and $R_q(x_j)$ are neighbors then $(\mathfrak{A}, R_p(x_i), R_q(x_j)) \equiv (\mathfrak{A}, R_p(y_i), R_q(y_j))$. Easily, if $\langle x_0, \ldots, x_m \rangle$ and $\langle y_0, \ldots, y_m \rangle$ are equivalent then they satisfy the same quantifier-free formulas in $\mathfrak{A}$, and $R_p(x_i)$ and $R_q(x_j)$ are neighbors iff so are $R_p(y_i)$ and $R_q(y_j)$.

So suppose $\langle x_0, \ldots, x_m \rangle$ and $\langle y_0, \ldots, y_m \rangle$ are equivalent, and suppose that $\{x_i, x_j\}$ is a prenucleus of $\{x_0, \ldots, x_m\}$ for a as demonstrated by the neighbors $R_p(x_i)$ and $R_q(x_j)$. Thus there is some $b \in A$ such that $(\mathfrak{A}, R_p(x_i), R_q(x_j), a) \equiv (\mathfrak{A}, R_p(y_i), R_q(y_j), b)$. Now it follows from Lemmas 1.1 and 1.2 and the Remark that $\langle x_0, \ldots, x_m, a \rangle$ and $\langle y_0, \ldots, y_m, b \rangle$ are equivalent. Continuing in a back-and-forth manner, we can build an automorphism $f: \mathfrak{A} \to \mathfrak{A}$ such that $f(x_i) = y_i$ for $i \leqslant m$ and $f(a) = b$. Thus $\langle x_0, \ldots, x_m, a \rangle$ and $\langle y_0, \ldots, y_m, b \rangle$

realize the same type, so that $\{x_i, x_j\}$ is indeed a nucleus of $\{x_0, \ldots, x_m\}$ for a. □

This proves Theorem 1. We can easily get a slight strengthening of this theorem.

COROLLARY 1.4. *If* $\mathfrak{A} = (A, <, R_0, \ldots, R_{n-1}, U_0, \ldots, U_{m-1})$ *is* $\aleph_0$*-categorical, where* $(A, <)$ *is a linearly ordered set, each* R_i *is monotone and each* $U_j \subseteq A$, *then* $\mathrm{Th}(\mathfrak{A})$ *is finitely axiomatizable.*

PROOF. For each $j < m$, let

$$S_j = \{(x,y) \in A \times A \colon x \leqslant y, \text{ and if } y \in U_j, \text{ then } x \neq y\}.$$

Then each S_j is a monotone realtion which is definable in $\mathfrak{A}$, and each U_j is definable in $(A, <, R_0, \ldots, R_{n-1}, S_0, \ldots, S_{m-1})$. Apply Theorem 1. □

2. **Partially ordered sets of width** 2. In this section we prove the main result.

THEOREM 2. *If* $\mathfrak{A} = (A, <)$ *is an* $\aleph_0$*-categorical, partially ordered set of width* 2, *then* $\mathrm{Th}(\mathfrak{A})$ *is finitely axiomatizable.*

First, we introduce some notation which will apply to any partially ordered set $(B, <)$. If $x, y \in B$, then let $x|y$ denote that x and y are incomparable (i.e., neither $x \leqslant y$ nor $y \leqslant x$). For $k < \omega$, let E_k be the binary relation such that

$$E_k(x,y) \leftrightarrow \exists x_0, \ldots, x_k(x = x_0|x_1|\cdots|x_k = y),$$

and let E be such that $E(x,y) \leftrightarrow \exists k E_k(x,y)$. Notice that E is an equivalence relation on B. Each E_k is definable in $(B, <)$, but in general E is not. However, if $(B, <)$ is $\aleph_0$-categorical, then a consequence of Ryll-Nardzewski's Theorem is that there is some n such that $E(x,y)$ iff $E_k(x,y)$ for some $k \leqslant n$. Thus if $(B, <)$ is $\aleph_0$-categrocial, then E is definable. We call the equivalence classes of E *components*, and say that $(B, <)$ is *simple* if B itself is a component.

Now we prove the theorem in the special case that, in addition to the given hypotheses, $\mathfrak{A}$ is simple.

Let $a \in A$ and define

$$A_0 = \{x \in A \colon \mathfrak{A} \vDash E_k(a,x) \text{ for some even } k \leqslant n\},$$

$$A_1 = \{x \in A \colon \mathfrak{A} \vDash E_k(a,x) \text{ for some odd } k \leqslant n\}.$$

It is easy to check that A_0 and A_1 are linearly ordered subsets of A and that $A_0 \cup A_1 = A$ and $A_0 \cap A_1 = \varnothing$. Define $\prec$ on A so that

$$x \prec y \leftrightarrow x < y \vee (x|y \wedge x \in A_0 \wedge y \in A_1),$$

and for $e = 0, 1$ define the binary relation R_e so that

$$R_e(x,y) \leftrightarrow \forall x_1, y_1((x_1 \preccurlyeq x \wedge y \preccurlyeq y_1 \wedge x_1 \in A_e) \rightarrow x_1 < y_1).$$

It is clear that $\prec$ linearly orders A, and that R_0 and R_1 are monotone relations (with respect to $(A, \prec)$). Each of $A_0, A_1, \prec, R_0$ and R_1 is definable in $(A, <, a)$. Conversely, the relation $<$ is definable in $(A, \prec, R_0, R_1, A_0, A_1)$ by

$$x < y \leftrightarrow (x \in A_0 \wedge R_0(x,y)) \vee (x \in A_1 \wedge R_1(x,y)) \vee ((x \in A_0 \leftrightarrow y \in A_0) \wedge x \prec y).$$

Now, since $\mathfrak{A}$ is $\aleph_0$-categorical, so is $(A, <, a)$, and hence also is $(A, \prec, R_0, R_1, A_0, A_1)$. But by Corollary 1.4, $\mathrm{Th}((A, \prec, R_0, R_1, A_0, A_1))$ is finitely axiomatizable; therefore, so is $\mathrm{Th}(\mathfrak{A})$. This proves the theorem for simple $\mathfrak{A}$.

Now consider any arbitrary $\aleph_0$-categorical $\mathfrak{A}$ of width 2, and consider the relation E on A. If X is a component, then $\mathfrak{A}|X$ is simple. It is easy to see that if X, Y are different components and $x \in X, y \in Y$, then either $x < y$ or $y < x$. If, in addition, $x_1 \in X$ and $y_1 \in Y$, then $x_1 < y_1$ iff $x < y$. Thus, there is an induced linear order $<$ on the set of components. By Ryll-Nardzewski's Theorem, there are components $X_0, \ldots, X_m$ such that if Y is any component, then $\mathfrak{A}|Y \equiv \mathfrak{A}|X_j$ for some $j \leqslant m$. By the first part of this proof, $\mathrm{Th}(\mathfrak{A}|X_j)$ is finitely axiomatizable for each $j \leqslant m$. Now let B be the set of components, and let $U_j = \{Y \in B\colon Y \equiv X_j\}$. Then $\mathfrak{B} = (B, <, U_0, \ldots, U_m)$ is $\aleph_0$-categorical, so that by Rosenstein's result [4] (or Corollary 1.4), $\mathrm{Th}(\mathfrak{B})$ is finitely axiomatizable. Now it is an easy matter to see how to recover $\mathfrak{A}$ from the structures $\mathfrak{A}|X_0, \ldots, \mathfrak{A}|X_m$ and $\mathfrak{B}$. Hence, it can be inferred on general grounds (e.g. Fefferman and Vaught [3]), or shown directly, that $\mathrm{Th}(\mathfrak{A})$ is finitely axiomatizable. □

3. **Comments on the proof.** An analysis of the proof of Theorem 2 reveals that $\mathrm{Th}(\mathfrak{A})$, for $\mathfrak{A}$ an $\aleph_0$-categorical, partially ordered set of width 2, is 2-nuclear. To see this, let E_x be the component of A containing x, and let $\prec_x$ be the linear order on E_x that x induces. Now suppose that $a \in A$, and that $X \subseteq A$ is nonempty and finite. Then there is a $Y \subseteq X$ consisting of at most 2 elements such that

(1) there is $y \in Y$ such that for all $x \in X \cap E_a$, if $x \preccurlyeq_a a$, then $x \preccurlyeq_a y \preccurlyeq_a a$;

(2) there is $y \in Y$ such that for all $x \in X$, if $E_x \leqslant E_a$, then $E_x \leqslant E_y \leqslant E_a$;

(3) the "duals" of (1) and (2) are true.

Then Y is a nucleus of X for a.

Another observation concerning the proof. Suppose that $X_0, \ldots, X_m$ are components of $\mathfrak{A}$ such that for any component Y there is a unique $j \leqslant m$ such that $\mathfrak{A}|Y \equiv \mathfrak{A}|X_j$. For each $j \leqslant m$, let p_j be a 1-type realized by some element in X_j. For each component Y let $a_Y \in Y$ be such that if $\mathfrak{A}|Y \equiv \mathfrak{A}|X_j$, then a_Y realizes the type p_j. Let

$$A_0 = \{x \in A\colon \mathfrak{A} \models E_k(a_Y, x) \text{ for some even } k \text{ and some component } Y\},$$

$$A_1 = \{x \in A\colon \mathfrak{A} \models E_k(a_Y, x) \text{ for some odd } k \text{ and some component } Y\}.$$

Then, as before, A_0 and A_1 are linearly ordered subsets of A such that $A_0 \cup A_1 = A$ and $A_0 \cap A_1 = \varnothing$. It is not hard to check that, in addition, $(\mathfrak{A}, A_0, A_1)$ is $\aleph_0$-categorical.

The previous discussion recalls the fundamental theorem of Dilworth [2] concerning partially ordered sets of width n. Dilworth's Theorem asserts that if a partially ordered set $(A, <)$ has width n, then A can be partitioned into n chains $A_0, \ldots, A_{n-1}$. We have seen that in the case $n = 2$, if we start with an $\aleph_0$-categorical partially ordered set, then we can find these chains so as to preserve $\aleph_0$-categoricity. This suggests the following natural question.

Question 3.1. If $(A, <)$ is an $\aleph_0$-categorical, partially ordered set of width n, do there exist chains $A_0, \ldots, A_{n-1}$ such that $A = A_0 \cup \cdots \cup A_{n-1}$ and $(A, <, A_0, \ldots, A_{n-1})$ is $\aleph_0$-categorical?

An affirmative answer to this question would imply that every $\aleph_0$-categorical, partially ordered set of finite width has a decidable theory. This is a consequence of the following proposition.

PROPOSITION 3.2. *If* $\mathfrak{A} = (A, <, A_0, \ldots, A_{n-1})$ *is an* $\aleph_0$*-categorical structure such that* $(A, <)$ *is a partially ordered set,* $A_0, \ldots, A_{n-1}$ *are chains, and* $A = A_0 \cup \cdots \cup A_{n-1}$, *then* $\mathrm{Th}(\mathfrak{A})$ *is finitely axiomatizable.*

PROOF. We can assume that $i < j < n$ implies that $A_i \cap A_j = \varnothing$. Define $\prec$ on A by

$$x \prec y \leftrightarrow \exists i, j(x \in A_i \wedge y \in A_j \wedge (j \leqslant i \rightarrow i = j \wedge x < y)).$$

For $i, j < n$ define R_{ij} by

$$R_{ij}(x,y) \leftrightarrow \forall x_1, y_1((x_1 \preccurlyeq x \wedge y \preccurlyeq y_1 \wedge x_1 \in A_i \wedge y_1 \in A_j) \rightarrow x_1 < y_1).$$

Each R_{ij} is monotone with respect to $(A, \prec)$, so that by Corollary 1.4, $\mathfrak{A}' = (A, \prec, A_0, \ldots, A_{n-1}, R_{ij})_{i,j<n}$ has a finitely axiomatizable theory. But $<$ is definable in $\mathfrak{A}'$, so that $\mathfrak{A}$ also has a finitely axiomatizable theory. □

4. **Epilogue.** Let T_n be the theory of partially ordered sets of width $\leqslant n$. The theory T_2, in spite of Theorem 2, is quite a bit more complicated than the theory T_1 of linearly ordered sets. For, it can be shown that T_2 is undecidable, whereas, as is well known, T_1 is decidable. We will spare the reader the details, although it is not difficult to see how to encode into T_2 the print-out of any Turing machine. Then for any r.e. set X, we can get a recursive sequence $\sigma_0, \sigma_1, \sigma_2, \ldots$ of sentences such that for each $n < \omega$, all of the following are equivalent:

(1) $n \in X$;

(2) $T_2 \cup \{\sigma_n\}$ is complete;

(3) $T_2 \cup \{\sigma_n\}$ has a finite model.

Since every finite model of T_2 is discrete, and no infinite discrete model of T_2 is $\aleph_0$-categorical, we see that we can include a fourth condition equivalent to (1)–(3):

(4) $T_2 \cup \{\sigma_n\}$ is $\aleph_0$-categorical.

Thus, it follows that the set of $\aleph_0$-categorical sentences σ consistent with T_2 is not recursive. This contrasts with the fact (see [1]) that the set of $\aleph_0$-categorical sentences consistent with T_1 is recursive. It is not clear whether or not the set of $\aleph_0$-categorical sentences consistent with T_2 is acutally r.e. However, a consequence of the nuclearity is that the union of this set and the set of all sentences inconsistent with T_2 is r.e.

References

1. R. Amit and S. Shelah, *The complete finitely axiomatized theories of order are dense*, Israel J. Math. **23** (1976), 200–208.

2. R. P. Dilworth, *A decomposition theorem for partially ordered sets*, Ann. of Math. (2) **51** (1950), 161–166. MR **11**, 309.

3. S. Feferman and R. L. Vaught, *The first order properties of products of algebraic systems*, Fund. Math. **47** (1959), 57–103. MR **21** #7171.

4. J. Rosenstein, *$\aleph_0$-categoricity of linear orderings*, Fund. Math. **64** (1969), 1–5. MR **39** #3982.

5. C. Ryll-Nardzewski, *On the categoricity in power $\leqslant \aleph_0$*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **7** (1959), 545–548. MR **22** #2543.

6. J. H. Schmerl, *On $\aleph_0$-categoricity and the theory of trees*, Fund. Math. (to appear).

7. ———, *$\aleph_0$-categoricity of partially ordered sets of width* 2, Notices Amer. Math. Soc. **23** (1976), A-496. Abstract #76T-E51.

Department of Mathematics, University of Connecticut, Storrs, Connecticut 06268