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STRONGLY CONTINUOUS SEMIGROUPS, WEAK SOLUTIONS, AND THE VARIATION OF CONSTANTS FORMULA

J. M. BALL

ABSTRACT. Let A be a densely defined closed linear operator on a Banach space X, and let $f \in L^1(0,\tau;X)$. A definition of weak solutions of the equation $\dot{u} = Au + f(t)$ is given. It is shown that a necessary and sufficient condition for the existence of unique weak solutions for every initial data in X is that A generate a strongly continuous semigroup on X, and that in this case the solution is given by the variation of constants formula.

Let A be a densely defined closed linear operator on a real or complex Banach space X, let $\tau > 0$ and let $f \in L^1(0,\tau;X)$. Let $D(A) \subseteq X$ denote the domain of A. It is well known (cf. Kato [3, p. 486]) that if A is the generator of a strongly continuous semigroup of bounded linear operators $\{T(t)\}, t \geqslant 0$, on X, and if $x \in D(A), f \in C([0,\tau]; X)$, then the equation

$$\dot{u}(t) = Au(t) + f(t), \qquad t \in (0,\tau], \tag{1}$$

has a unique continuous solution satisfying $u(0) = x$, and that u is given by

$$u(t) = T(t)x + \int_0^t T(t-s)f(s)\,ds, \qquad t \in [0,\tau]. \tag{2}$$

When $x \in X$ is arbitrary, then unless $\{T(t)\}$ and f have special properties (e.g. $\{T(t)\}$ holomorphic and f Hölder continuous), $u(t)$ given by (2) will not, in general, belong to $D(A)$ for $t \in (0,\tau]$, so that (1) does not even make sense. The purpose of this note is to establish an abstract equivalence between functions u given by (2) and weak solutions, suitably defined, of (1), and to give a related characterization of strongly continuous semigroups. Although the proof of the theorem is simple, there seems to be no statement of it in the literature even in the case $f = 0$. An application to a class of nonlinear operator equations including certain nonlinear wave equations appears in [1].

Let A^* denote the adjoint of A and $\langle\, , \rangle$ the pairing between X and its dual space X^*.

DEFINITION. A function $u \in C([0,\tau]; X)$ is a *weak solution* of (1) if and only if for every $v \in D(A^*)$ the function $\langle u(t),v\rangle$ is absolutely continuous on $[0,\tau]$ and

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$$\frac{d}{dt}\langle u(t),v\rangle = \langle u(t),A^*v\rangle + \langle f(t),v\rangle \tag{3}$$

for almost all $t \in [0,\tau]$.

THEOREM. *There exists for each $x \in X$ a unique weak solution $u(t)$ of* (1) *satisfying $u(0) = x$ if and only if A is the generator of a strongly continuous semigroup $\{T(t)\}$ of bounded linear operators on X, and in this case $u(t)$ is given by* (2).

We need the following lemma (cf. Goldberg [2, p. 127]).

LEMMA. *Let x, $z \in X$ satisfy $\langle z,v\rangle = \langle x,A^*v\rangle$ for all $v \in D(A^*)$. Then $x \in D(A)$ and $z = Ax$.*

PROOF. Let $G(A) \subseteq X \times X$ denote the graph of A, which is closed by assumption. By the Hahn-Banach theorem there exist v, $v^* \in X^*$, such that $\langle Ax,v\rangle + \langle x,v^*\rangle = 0$ for all $x \in D(A)$, and $\langle z,v\rangle + \langle x,v^*\rangle \neq 0$. Thus $v \in D(A^*)$, $v^* = -A^*v$ and $\langle z,v\rangle \neq \langle x,A^*v\rangle$, which is a contradiction.

PROOF OF THEOREM. Let A generate the strongly continuous semigroup $\{T(t)\}$. There exists a constant M such that $\|T(t)\| \leqslant M$ for $t \in [0,\tau]$. First note that if $x \in X$ and $v \in D(A^*)$ then $\langle T(t)x,v\rangle$ is differentiable with respect to t with derivative $\langle T(t)x,A^*v\rangle$. This is obvious if $x \in D(A)$, and holds for arbitrary $x \in X$ because $D(A)$ is dense and $\{T(t)\}$ strongly continuous. Let u be given by (2). It is easily shown that $u \in C([0,\tau];X)$. For every $v \in D(A^*)$ and $t \in [0,\tau]$,

$$\langle u(t),v\rangle = \langle T(t)x,v\rangle + \int_0^t \langle T(t-s)f(s),v\rangle\, ds.$$

Suppose that $f \in C([0,\tau];X)$. Since $(t,x) \mapsto T(t)x$ is jointly continuous on $[0,\tau] \times X$ it follows that

$$\frac{d}{dt}\int_0^t \langle T(t-s)f(s),v\rangle\, ds = \langle f(t),v\rangle + \int_0^t \langle T(t-s)f(s),A^*v\rangle\, ds,$$

so that $\langle u(t),v\rangle$ is differentiable for $t \in [0,\tau]$ and satisfies (3). If $f \in L^1(0,\tau;X)$, let $f_n \in C([0,\tau];X)$ for $n = 1,2,\ldots,$ with $f_n \to f$ in $L^1(0,\tau;X)$ and define

$$u_n(t) = T(t)x + \int_0^t T(t-s)f_n(s)\, ds, \qquad s \in [0,\tau].$$

Then

$$\|u_n(t) - u(t)\| \leqslant M\int_0^\tau \|f_n(s) - f(s)\|\, ds,$$

so that $u_n \to u$ in $C([0,\tau];X)$. But by the above, for each $v \in D(A^*)$,

$$\langle u_n(t),v\rangle = \langle x,v\rangle + \int_0^t \left[\langle u_n(s),A^*v\rangle + \langle f_n(s),v\rangle\right] ds, \qquad t \in [0,\tau].$$

Passing to the limit we see that u is a weak solution of (1).

Next we prove that $u(t)$ is the only weak solution of (1) satisfying $u(0) = x$. Let $\bar{u}(t)$ be another such weak solution and set $w = u - \bar{u}$. Then

$$\langle w(t),v\rangle = \left\langle \int_0^t w(s)\,ds, A^*v\right\rangle$$

for all $v \in D(A^*)$, $t \in [0,\tau]$, so that by the lemma, $z(t) \stackrel{\text{def}}{=} \int_0^t w(s)\,ds$ belongs to $D(A)$ and $\dot{z} = Az$. By [3, p. 481] $z = 0$ and hence $u = \bar{u}$.

Suppose that A is such that (1) has, for each $x \in X$, a unique weak solution $u(t)$ satisfying $u(0) = x$. For $t \in [0,\tau]$ define $T(t)x = u(t) - u_0(t)$, where u_0 is the weak solution of (1) satisfying $u_0(0) = 0$. If $t \geqslant 0$ let $t = n\tau + s$, where n is a nonnegative integer and $s \in [0,\tau)$, and define $T(t)x = T(s)T(\tau)^n x$. The map $\theta\colon X \to C([0,\tau];X)$ defined by $\theta(x) = T(\cdot)x$ has closed graph and, hence, $T(\cdot)$ is a strongly continuous semigroup. Let B be the generator of $T(\cdot)$ and let $x \in D(B)$. For any $v \in D(A^*)$,

$$\frac{d}{dt}\langle T(t)x,v\rangle\Big|_{t=0} = \langle Bx,v\rangle = \langle x,A^*v\rangle.$$

It follows from the lemma that $x \in D(A)$ and $Bx = Ax$. In particular, $D(B) \subseteq D(A)$. The proof of the theorem is completed by showing that $D(A) \subseteq D(B)$.

Let $x \in D(A)$. Using the lemma we see that for each $t \in [0,\tau]$ the integrals $\int_0^t T(s)x\,ds$ and $\int_0^t T(s)Ax\,ds$ belong to $D(A)$ and

$$T(t)x = x + A\int_0^t T(s)x\,ds, \tag{4}$$

$$T(t)Ax = Ax + A\int_0^t T(s)Ax\,ds. \tag{5}$$

Consider the function

$$z(t) = \int_0^t T(s)Ax\,ds - A\int_0^t T(s)x\,ds.$$

It follows from (4) that $z \in C([0,\tau];X)$. Clearly $z(0) = 0$. Let $v \in D(A^*)$. Using (4) and (5) we see that

$$\frac{d}{dt}\langle z(t),v\rangle = \langle z(t),A^*v\rangle, \qquad t \in [0,\tau].$$

But it follows from our assumptions that the equation $\dot{z} = Az$, $z(0) = 0$, has only the zero weak solution. Hence

$$\int_0^t T(s)Ax\,ds = A\int_0^t T(s)x\,ds, \qquad t \in [0,\tau].$$

Therefore by (4),

$$\lim_{t\to 0+} \frac{1}{t}\left[T(t)x - x\right] = Ax$$

and, hence, $x \in D(B)$.

Note added in proof. The 'if' part of the above theorem is stated and proved by a somewhat different method in the recent book by Balakrishnan

[4, Theorem 4.8.3] under the assumption that X is a Hilbert space.

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Department of Mathematics, Heriot-Watt University, Riccarton, Currie, Edinburgh EH14 4AS, Scotland