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LINEAR REPRESENTATIONS OF SEMIGROUPS OF BOOLEAN MATRICES

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ABSTRACT. Let B_n be the multiplicative semigroup of $n \times n$ matrices over the semiring 0, 1 under the operations "or" and "and". We show that the least possible degree of a faithful representation of B_n over a field is $2^n - 1$ by studying representations of a subsemigroup of B_n. By different methods we answer the same question for the subsemigroups of Boolean matrices greater than or equal to some permutation matrix (Hall matrices) and greater than or equal to the identity (reflexive Boolean matrices). We prove every representation of the latter semigroup can be triangularized.

Let B_n be the semigroup of $n \times n$ matrices over the semiring formed by 0,1 under the operations ab and $\sup(a, b)$. We call such matrices *Boolean matrices* and corresponding n-tuples on which they act, *Boolean vectors*. Since there are $2^n - 1$ nonzero Boolean vectors of dimension n, action of B_n on Boolean vectors gives a homomorphism from B_n into the semigroup of partial transformations PT_{2^n-1} (which is the semigroup of maps from $\{0, 1, \ldots, 2^n - 1\}$ to itself sending 0 to 0). This gives a matrix representation of degree $2^n - 1$ over any field, by matrices which have at most one nonzero entry in each row. We show that there are no faithful representations of B_n of lower degree over a field by studying representations of a semigroup of B_n.

We consider the same question for the subsemigroups of B_n consisting of Hall matrices and reflexive matrices [1], [3].

DEFINITION. The *semigroup of reflexive Boolean matrices* is the semigroup of all Boolean matrices $B = (b_{ij})$ such that $b_{ii} = 1$.

DEFINITION. The *semigroup of Hall matrices* is the semigroup of Boolean matrices B such that some subset of the "1" entries of B is the set of "1" entries of a permutation matrix.

In these cases a faithful quotient of the representation above, of dimension $2^n - 2$, can be obtained by sending the basis element corresponding to the Boolean vector $(1, 1, \ldots, 1)$ to 0. We show that $2^n - 2$ is the least possible dimension of a faithful representation of either of these semigroups, and establish a slightly stronger result for Hall matrices. We also note that every

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representation of the semigroup of reflexive Boolean matrices over a field can be triangularized.

We study B_n by means of a subsemigroup first investigated by Petrich [2].

DEFINITION. A Boolean matrix $A = (a_{ij})$ is a *cross vector* iff there exist Boolean vectors v, w such that $a_{ij} = v_i w_j$ for all i and j, where v_k is the kth element of v.

We frequently denote such a matrix as (v, w). Composition of two cross vectors $(v,w)(x,y) = (w \cdot x)(v,y)$ where $w \cdot x$ is a Boolean inner product $\sup_i(w_i x_i)$. Petrich [2] observed that the semigroup of cross vectors is a regular Rees matrix semigroup of degree $2^n - 1$. In fact if $G = (e)$, and I, Λ are each the set of nonzero Boolean vectors, $M^0(G, I, \Lambda\ P)$ is the semigroup of cross vectors where P has a one in location v, w iff $v \cdot w = 1$.

Let $\mathbf{Z}C$ denote the semigroup ring of the semigroup C of cross vectors (with coefficients in $\mathbf{Z}$). Let $\mathbf{Z}C_0$ denote the factor ring obtained by adjoining to $\mathbf{Z}C$ a relation equating the zero matrix to 0. Let h_1 be the homomorphism $C \to PT_{2^n-1}$ given by partial transformations on the nonzero Boolean vectors and 0. Let h_2: $PT_{2^n-1} \to M_{2^n-1}(\mathbf{Z})$ ($(2^n - 1) \times (2^n - 1)$ matrices over $\mathbf{Z}$) be the representation by means of matrices with at most one nonzero entry in each row, all entries being 0 or 1.

THEOREM 1. *The map $\mathbf{Z}C_0 \to M_{2^n-1}(\mathbf{Z})$ induced by $h_2 h_1$: $C \to M_{2^n-1}(\mathbf{Z})$ is an isomorphism.*

PROOF. It suffices to show that this map is an epimorphism. Let a basis u_w for the space of vectors of integers $\mathbf{Z}^{2^n-1}$ on which M_{2^n-1} acts be chosen in 1-1 correspondence with nonzero Boolean vectors w. Let $x_{(v,w)}$ denote the basis element of $\mathbf{Z}C_0$ corresponding to the cross vector (v, w). A sum $\Sigma a_v x_{(v,w)}$ acts by sending u_z to $(\Sigma a_v(z \cdot v))w$. Therefore all matrices of $M_{2^n-1}(\mathbf{Z})$ will be in the image of $\mathbf{Z}C_0$ if the linear forms $T_a(z) = \Sigma_v a_v(z \cdot v)$ range over all integer-valued linear forms, in particular, if the forms f_y sending u_y to 1 and all other u_w to 0 are given by T_a for some choice of integers a_v.

Suppose, on the contrary, that y is a Boolean vector having the fewest possible number of 1's such that f_y is not equal to any form T_a. Let y' denote the vector having 1's where y has 0's and 0's where y has 1's. Let 1 denote the Boolean vector $(1, \ldots, 1)$. The form $T_a(z) = (z \cdot y') - (z \cdot 1)$ is 1 on all Boolean vectors which are $\leqslant z$ and 0 on all others. By the minimality of y, the form which is 1 on all Boolean vectors $< z$ and 0 on all others can be obtained as some T_b. Therefore $T_a - T_b$ represents f_y as a linear form $\Sigma_v a_v(z \cdot v)$, a contradiction. This proves the theorem.

COROLLARY. *For any field k, $kC_0 \simeq M_{2^n-1}(k)$ and $kC \simeq k \times M_{2^n-1}(k)$.*

COROLLARY. *Every nonzero representation of the semigroup of cross vectors over a field has degree at least $2^n - 1$, and so does every faithful representation of the semigroup of Boolean matrices B_n.*

The next two theorems, and their preceding lemmas, were suggested by George Bergman.

DEFINITION. The *standard representation* of the semigroup of Hall matrices or the semigroup of reflexive Boolean matrices is the quotient of h_2h_1 obtained by setting the basis element u_1 equal to 0. Here u_1 is the basis element corresponding to the Boolean vector $(1, \ldots, 1)$.

DEFINITION. Let R be a ring and X a subset of R. Then an R-module M will be called *X-faithful* if no element of X annihilates all of M.

The set of elements of X annihilating $a \in M$ will be denoted $\mathrm{Ann}_X(a)$. Modules will be right modules. For $R = kS$ for some semigroup S, a faithful representation of S is an X-faithful representation of R where $X = \{s - t\colon s, t \in S, s \neq t\}$.

LEMMA 2. *Let R be a ring and X a subset of R. Suppose I is a right ideal of R such that*

(a) *I is X-faithful as a right module.*

(b) *I has a least nonzero subideal J.*

(c) *J has nonempty intersection with X. Then an R-module M is X-faithful iff M has a submodule isomorphic to I.*

PROOF. $\Rightarrow$ Assumption (a).
$\Leftarrow$ Choose an element $x \in X \cap J$. Since M is assumed faithful, $Mx \neq 0$. Let a be such that $ax \neq 0$. Then $aI \subseteq M$ will be isomorphic to I, since the kernel of multiplication by a will be a right subideal of I thus containing J if it is nonzero. But this subideal does not contain x.

THEOREM 3. *Any faithful linear representation of the semigroup of Hall matrices over a field k has a subrepresentation isomorphic to the standard representation.*

PROOF. Let $R = kS$, S denoting the semigroup of Hall matrices. We map the standard representation into R by sending u_z to $p_z - p_1$, where p_z is the matrix having z for its first row and 1 for all other rows. This is a monomorphism of right R-modules, so its image is a right ideal I of R.

We will show that the R-submodule of the standard representation spanned by the elements u_z where z ranges over the Boolean vectors with one 0 is the least nonzero submodule. Let N be any submodule and let $\Sigma u_v \alpha_v$ be any nonzero element of N, where the α_v are elements of k. Among the elements u_v occurring with nonzero coefficient α_v choose one u_w for which w has the fewest 1's. Let $s \in S$ denote the matrix whose ith row is $[0, 1, \ldots, 1]$ if $w_i = 1$ and 1 if $w_i = 0$. Then s carries a nonzero Boolean vector z to $[0, 1, \ldots, 1]$ if $z \leqslant w$ and to 1 otherwise. Hence $s\alpha_w^{-1}$ will carry $\Sigma u_v \alpha_v$ to $u_{[0,1,\ldots,1]}$ so N contains this element. Applying permutation matrices we see that for every z with only one 0, $u_z \in N$, so the submodule spanned by these elements is contained in N, and so is the least nonzero submodule.

Under the embedding of the standard representation in R, the element $u_{[0,1,\ldots,1]}$ by definition does go to an element of the form $s - t$. So the lemma applies and proves the theorem.

LEMMA 4. *Let R be an algebra over a field k and X a subset of R. Suppose R has a subset U such that*:

(a) *Any two elements of U have distinct right X-annihilators.*

(b) *For each $u \in U$ there exists $x \in X$ such that $ux \neq 0$ but $vx = 0$ for every $v \in U$ such that* $\mathrm{Ann}_X v \not\subseteq \mathrm{Ann}_X u$.

Then every $(UX - \{0\})$-faithful right R-module M has k-dimension at least $|U|$.

PROOF. Let M be a $(UX - \{0\})$-faithful right R-module.

For each $u \in U$ choose $x_u \in X$ satisfying the condition of (b). Since ux_u is nonzero and M is $(UX - \{0\})$-faithful, we can find $a_u \in M$ with $a_u u x_u \neq 0$. We will show that the set $\{a_u u\colon u \in U\}$ is a k-linearly independent subset of M. Consider any nontrivial k-linear combination

$$\sum_{U_0} a_u u \alpha_u \tag{1}$$

where U_0 is a finite nonempty subset of U and α_u is a nonzero element of k for each $u \in U_0$. Let u_0 be such that $\mathrm{Ann}_X(u_0)$ is minimal among the u's in U_0. Then all elements $v \neq u_0$, $v \in U_0$, are annihilated by x_{u_0} by (b) and (a). So multiplying (1) by $x_{u_0}\, \alpha_{u_0}^{-1}$ we get $a_{u_0} u_0 x_{u_0}$ which is nonzero by the choice of a's. So (1) is nonzero which proves the linear independence of $\{a_u u\}$.

THEOREM 5. *Any faithful linear representation of the semigroup S of $n \times n$ reflexive matrices has dimension $\geqslant 2^n - 2$.*

PROOF. Let $R = kS$.

It is no longer true that the standard representation is isomorphic to a right ideal of R. However for each basis vector u_z of the standard representation there exists a right ideal of R isomorphic to the submodule $u_z R$ of the standard representation. Indeed, the submodule $u_z R$ is spanned by u_w as w ranges over all Boolean vectors $\geqslant z$. For such w define p_w^z to be the matrix whose ith row is w if $z(i) = 1$ and is 1 if $z(i) = 0$. Then the linear map sending u_w to $p_w^z - p_1^z$ is an embedding of right R-modules.

Let $U = X = \{p_z^z - p_1^z\colon z$ is a Boolean vector other than 0 or $1\}$. An element of UX will have the form

$$(p_z^z - p_1^z)(p_w^w - p_1^w) = p_z^z p_w^w - p_1^z p_1^w$$

(because $p_1^z p_w^w = p_z^z p_1^w =$ the matrix consisting entirely of 1's). So if it is nonzero, it will lie in $\{s - t\colon s \neq t,\ s, t \in S\}$. So any faithful representation of S will also be a $(UX - \{0\})$-faithful R-module.

We now show that $\{u_z\}$, the basis of the standard representation, has properties (a) and (b) of the preceding lemma. Since the module $u_z R$ is isomorphic to the ideal generated by $p_z^z - p_1^z$, $\mathrm{Ann}_X\{u_z\} = \mathrm{Ann}_X\{p_z^z - p_1^z\}$.

Thus the properties (a), (b) for the $\{u_z\}$ imply (a), (b) for U and the lemma will imply any faithful representation of S has dimension at least $|U| = 2^n - 2$.

For Boolean vectors, z, w other than 0, 1 we have $u_z(p_w^w - p_1^w) \neq 0$ iff $z \leqslant w$. This means that $\text{Ann}_X v = \{p_w^w - p_1^w: w \ngeqslant v\}$. So $\text{Ann}_X u \supseteq \text{Ann}_X v$ iff $u \geqslant v$. Condition (a) follows immediately, and condition (b) follows on taking $x = p_u^u - p_1^u$. This proves the theorem.

DEFINITION. The $\mathfrak{R}$-*ordering* in a semigroup is the quasiorder $x \leqslant y$ iff x belongs to the principal right ideal generated by y. The $\mathfrak{R}$-classes are the equivalence classes under the equivalence relation $x\ \mathfrak{R}\ y$ iff x and y generate the same principal right ideal.

In the semigroup of reflexive Boolean matrices, the principal right ideal generated by a matrix A consists of A together with other matrices which are strictly larger than A as 0, 1 matrices. Thus the $\mathfrak{R}$-classes in this semigroup each contain only one element.

THEOREM 6. *Let S be a finite semigroup all of whose $\mathfrak{R}$-classes contain only one element. Then every finite dimensional representation of S over a field is equivalent to one in which all matrices representing elements of S are triangular.*

PROOF. It suffices to show that for every such representation of degree > 1 there will be a proper invariant subspace. For such a representation ρ, let x be an $\mathfrak{R}$-minimal element of the set of elements of S for which $\rho(x) \neq 0$. Then for all a in S, $\rho(x)\rho(a) = \rho(xa) = 0$ or $= \rho(x)$. Therefore the image of $\rho(x)$ is an invariant subspace. Suppose it is the entire space of the representation. Then for $\rho(a) \neq 0$, image $\rho(x)\rho(a) \neq 0$. Therefore for $\rho(a) \neq 0$, $\rho(x)\rho(a) = \rho(x)$. Therefore for all a, either $\rho(a) = 0$ or $\rho(a)$ is the identity matrix. Thus if degree $\rho > 1$ there will be a proper nonzero invariant subspace. This proves the theorem.

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