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PREFERRED SETS IN TOPOLOGICAL DYNAMICS

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ABSTRACT. The second author introduced a set of dynamical invariants called indivisibilities in one of his previous papers [3]. Following the same idea but from a different point of view this note concerns a class of invariants in transformation groups called the 'preferred sets'. For a special class of transformation groups, the number of permissible preferred sets is determined.

In Lam [3] a topological dynamical invariant $(\xi, Y, \mathfrak{Q}, \{K_i\})$ called *indivisibility* was studied. The symbols represent a transformation group $\xi = (X, T)$, a subset Y of X, the set of nets $\mathfrak{Q}$ in the acting group T and a family $\{K_i\}$ of subsets X. It is defined as follows: For every $K \in \{K_i\}$, whenever there exists a point $y_0 \in Y$ such that for some $\alpha \in \mathfrak{Q}$ the net $y_0\alpha$ has a limit point in K, then the net $y\alpha$ has a limit point in K for all $y \in Y$. In this paper we will consider the case where the set $\{K_i\}$ consists of only one set K. If K is nonempty, we call it a *preferred set* of Y. We will study the family of preferred sets of Y especially for the case that Y has a finite number of preferred sets. In this paper we will denote a transformation group by (X, T). The phase space X will be a T_2 uniform space with a prescribed uniformity. The group T is, in general, an arbitrary topological group (cf. [1],[2] for references).

DEFINITION. A subset K of X is said to be a *preferred set* of Y in X if K is nonempty and if there exists $y_0 \in Y$ such that $y_0\alpha$ has a limit point in K for a certain net α, then $y\alpha$ has a limit point in K for all $y \in Y$. If K does not contain any proper subset which is a preferred set of Y, then K is said to be a minimal preferred set (of Y).

If X is compact, we let E (not to be confused with $E(X)$, the set of all equicontinuous points of X) be the enveloping semigroup of X (see [1]). Then a nonempty set $K \subset X$ is a preferred set of $Y \subset X$ if and only if for every $f \in E$ either $f(Y) \cap K = \varnothing$ or $f(Y) \subset K$. If X is noncompact, a set Y need not have a preferred set. In fact if Y is the set $E(X)$ of Example 5.4 of Lam [3], and if the fixed point $(\frac{1}{2}, 0, 0)$ is removed from the phase space X, then the set $E(X)$ has no preferred sets. We begin with some lemmas.

LEMMA 1. *Let K and Y be subsets of X. The following property holds. If K is a preferred set of Y which intersects Y, then K contains Y.*

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Proof. We need only take α to be a net of identity elements in the definition of a preferred set.

The notion of preferred sets may be used to characterize the property that a set is $\mathfrak{Q}$3-indivisible studied in [3,3.1] (in notation $\mathfrak{Q} \mathrel{+\!\!\!+\!\!\!+} Y$). The third indivisibility was considered in [3] as an important property. It is now characterized by a few other properties, among them the following is one.

Definition. A set $Y \subset X$ is said to be *totally equitable* if for every decomposition of X if all but one of the sets are preferred sets of Y, then the last is also a preferred set of Y. Equivalently, Y is totally equitable if for every preferred set K of Y which is not X, its complement $X - K$ is also a preferred set of Y. For a subset Y of X the following properties hold. If $\{K_i\}$ is a nonempty family of preferred sets of Y, then $\bigcup_i K_i$ is a preferred set of Y, and if $\bigcap_i K_i \neq \varnothing$ then the intersection is also a preferred set of Y.

Lemma 2. *If K_1 and K_2 are two distinct subsets of X such that $K_2 \subset K_1$ and if both sets are preferred sets of Y, then the difference set $K_1 - K_2$ is also a preferred set of Y.*

Proof. The proof is actually very simple. Suppose the set $K_1 - K_2$ is not preferred. Then there exist two points $a, b \in Y$ and $\alpha \in \mathfrak{Q}$ such that $a\alpha$ has a limit point in $K_1 - K_2$ and $b\alpha$ does not have a limit point in $K_1 - K_2$. We may assume that $x\alpha$ converges to a point in a compactification of X, for instance, the one-point compactification of [3], for all $x \in Y$. Now K_1 is a preferred set of Y. Since $\lim a\alpha$ is in $K_1 - K_2$, we must also have that $\lim b\alpha \in K_1$. But we also have $\lim b\alpha \notin K_1 - K_2$, hence $\lim b\alpha \in K_2$. Since K_2 is a preferred set of Y, $\lim a\alpha \in K_2$, which contradicts that $\lim a\alpha \in K_1 - K_2$. Hence $K_1 - K_2$ is a preferred set of Y.

Lemma 3. *For $Y \subset X$ the following statements are pairwise equivalent.*

(1) *$\mathfrak{Q} \mathrel{+\!\!\!+\!\!\!+} Y$, i.e. X is a preferred set of Y.*

(2) *Y is totally equitable and it has at least one preferred set.*

(3) *The family of preferred sets of Y forms a covering of X.*

(4) *X is decomposed into a disjoint union of minimal preferred sets of Y.*

Proof. (1) $\Rightarrow$ (2). Suppose Y has a preferred set K such that $X - K \neq \varnothing$. According to Lemma 2 the set $X - K$ is a preferred set of Y. Hence Y is totally equitable.

(2) $\Rightarrow$ (3). Let $\{K_i\}$ be the nonempty family of all preferred sets of Y. The $\bigcup_i K_i$ is then a preferred set of Y. Since Y is totally equitable and $X - \bigcup_i K_i$ cannot be a preferred set, we have $X = \bigcup_i K_i$

(3) $\Rightarrow$ (4). Each $x \in X$ is contained in a preferred set, hence $M = \bigcap \{K \mid K$ a preferred set of Y contains $x\}$ is a preferred set. If M is not a minimal preferred set of Y, then there would be a preferred proper subset K_0 in M. Since $M - K_0$ is also a preferred set of Y, M is then either equal to K_0 or equal to $M - K_0$ which is a contradiction.

(4) $\Rightarrow$ (1). Obvious. The proof of Lemma 3 is completed.

As a by-product of considering Lemma 3, we have the following simple result.

THEOREM 1. *If the number of preferred sets of* $Y \subset X$ *is finite, then it is* $2^n - 1$, *where* $n = 0,1,2,\ldots$.

PROOF. Either Y has no preferred set or there exists a maximum preferred set R, which contains all other preferred sets. The proof of Lemma 3 shows that R is decomposed into a disjoint union of minimal preferred sets of Y. It follows easily that the number of preferred sets is $2^n - 1$. Easy examples show that all numbers $2^n - 1$ can be attained.

We now consider three theorems, which are the main results of the paper. The theorems can be applied to subsets Y of $E(X)$.

THEOREM 2. *For a transformation group* (X,T), *where* X *is a metric space which is locally compact and connected and* T *is also connected, let* $N(X)$, *the set of all nonequicontinuous points of* X, *be* 0-*dimensional. Then the number of preferred sets of* $E(X)$ *is either* 0,1,3 *or* 7.

PROOF. Since T is connected, the components of $E(X)$ are invariant. We take one of them and call it M. Let $F = \mathrm{Cl}(M)$; then F is closed and invariant. We consider the restricted transformation group (F,T) and denote by $E(F)$ its set of equicontinuous points. By Lemma 1 there exists at most one minimal preferred set of $E(X)$ overlapping $E(X)$. We claim that if $E(X)$ has a fixed point x_0, then $E(X)$ has at most one other minimal preferred set. Thus let K be another preferred set of $E(X)$; then $K \subset N(X)$. Let $p \in K$. By a lemma proved in Lam [4] (see Lemma 4 of this paper) there exists $a \in E(X)$ with a sequence α in T such that $\lim a\alpha = p$. We may assume that α has the property as that in the proof of Lemma 2, then we may define a function $f\colon E(X) \to X^*$, $f(x) = \lim x\alpha$, where X^* is the one-point compactification of X. Since K is a preferred set of $E(X)$, we have $f(M) \subset K$. It is clear that f is continuous and therefore $f(M)$ is a singleton, say q in the totally disconnected set K. Since $x_0 \in E(F)$, we have $\lim x_0\alpha = q$. However, x_0 is a fixed point, hence $x_0 = q$. Hence $x_0 \in K$. There is then at most one such set K. Hence $E(X)$ has 1 or 3 preferred sets in case there is a minimal preferred set. It remains to show the case when $E(F)$ has no fixed points. Since T is connected, the set $N(X)$ under the present assumption, consists of fixed points and since the set of fixed points is closed, we may assume $\mathrm{Cl}(N(X)) \cap E(F) = \varnothing$. If $N(F)$ is the set of nonequicontinuous points of (F,T), then

$$\mathrm{Cl}(N(X)) \cap F \subset N(F) \subset \mathrm{Cl}(N(X)) \cap F.$$

Hence

$$N(F) = N(X) \cap F = \mathrm{Cl}(N(X)) \cap F.$$

In particular, $N(F)$ is closed, and if it is not empty it is what is called a *strictly almost equicontinuous transformation group* (SAE) in Lam [4]. By definition a transformation group (X,T) is SAE if X is locally compact metric, $N(X)$ is 0-dimensional, $\mathrm{Cl}(pT)$ is compact if $p \in N(X)$ and $E(X)$ is completely indivisible by $\mathfrak{Q}$ (by that we mean $(\xi, E(X), \mathfrak{Q}, \mathfrak{K})$ are indivisibilities for (i) $\mathfrak{K}$ to be the set of components of $N(X)$ and for (ii) $\mathfrak{K}$ to be the single set X.) It was proved there that if $E(X)$ is connected and $N(X)$ is closed then the indivisibility conditions were satisfied. Theorem 1.17 in [4] states that for SAE the set $N(F)$ consists of none, one or two points. If K is a minimal preferred set of $E(X)$ and p is a point in it we can define a function $f\colon E(X) \to X^*$ and show that $p \in f(M) \subset K$. Since M is a subset of F which is closed and invariant we have $f(M) \subset F$. Hence $f(M) \subset K \cap F \subset N(X) \cap F = N(F)$. It follows that there can be at most two such sets K. It should be noted that under the hypothesis of the theorem, the set $N(X)$ can be infinite (see [3, Example 5.4]).

LEMMA 4. *Let X be a locally compact and connected metric space. Let $N(X)$ be nonempty, 0-dimensional and if $p \in N(X)$, then $\mathrm{Cl}(pT)$ is compact. Then for every 0-dimensional set $A \subset X$ any sequence $\{t_i\}$ which is nonequicontinuous at a point $p \in N(X)$ has a subsequence $\{t_{i_j}\}$ of $\{t_i\}$ and a point $q \in E(X) - A$ such that $\lim qt_{i_j}^{-1} = p$.*

PROOF. The case when $A = \varnothing$ is proved in Lemma 1.11 of [4]. A slight modification of the proof is sufficient for the present case.

THEOREM 3. *Let X again be a locally compact connected metric space, $N(X)$ 0-dimensional and $\mathrm{Cl}(pT)$ compact for all $p \in N(X)$. If T is abelian and $E(X)$ has an invariant component, then the number of preferred sets of $E(X)$ is 0,1,3 or 7.*

PROOF. Let R in Theorem 1 exist and let M be the invariant component of $E(X)$. Consider $F = \mathrm{Cl}(M)$. Let K_1 be a minimal preferred set of $E(X)$ which overlaps $E(X)$; in general K_1 exists. If there exists a minimal preferred set K_2 of $E(X)$ not intersecting $E(X)$, let $p \in K_2$. Then there exists a sequence α in T such that $\lim a\alpha = p$ for some $a \in E(X)$. We may assume that $f\colon E(X) \to X^*$, given by $f(y) = \lim y\alpha$, exists. Then $f(M)$ is a singleton $q \in K_2$, and $q \in R \cap (N(X) \cap F)$. If we can show that there are at most two points in $B = R \cap (N(X) \cap F) - K_1$, then there can be at most two such subsets K_2 of X.

Thus let there be three distinct points p_1, p_2 and p_3 in B. We first show that they are fixed points. Since the preferred sets of $E(X)$ cover B we may assume that $p_1 \in K_2$. Then $\lim x\alpha = p_1$ for all $x \in M_1 = E(X) \cap F$. The set M_1 is invariant and connected. Let $x_1 \in M_1$ and $t \in T$, then $x_1 t \in M_1$. Since T is abelian, we have

$$p_1 t = \lim x_1 \alpha t = \lim x_1 t \alpha = p_1.$$

Hence, p_1 and, similarly, p_2 and p_3 are fixed points. Since M_1 is dense in F, in particular, at p_2 and p_3 in (F,T), we see that the sequence α is nonequicontinuous at the fixed points p_2 and p_3. Let A in Lemma 4 be $E(F) - M_1$. By applying Lemma 4 twice on (F,T) we can find two points $a,b \in E(F) - A$

$= M_1$ and a sequence $\{t_{i_j}\}$ of $\{t_i\}$ such that $\lim at_{i_j}^{-1} = p_2$ and $\lim bt_{i_j}^{-1} = p_3$. However, it is easily seen that each of the singleton sets $\{p_2\}$ and $\{p_3\}$ is a preferred set of M_1. We then have a contradiction. Hence B has at most two points.

LEMMA 5. *Let A be a syndetic subgroup of T (i.e. there exists a compact set $K \subset T$ for the subgroup T such that $AK = T$). If* $\mathrm{Cl}(pT)$ *is compact for all $p \in N(X)$, then $N(X)$ is also the set of all nonequicontinuous points of A.*

PROOF. Straightforward argument.

We now state the theorem for the last case.

THEOREM 4. *Let X be a locally compact and connected metric space and let $N(X)$ be finite. Then the number of preferred sets of $E(X)$ is* 0,1,3 *or* 7.

PROOF. We may assume that $E(X)$ has at least one preferred set K. If $K \cap E(X) \neq \varnothing$, then by Lemma 1, $E(X) \subset K$. If so, then K may be assumed to be the minimal preferred set containing $E(X)$. Let K_1 be any other possible minimal preferred set of $E(X)$. Then $K_1 \subset N(X)$. Let $p \in K_1$. By Lam [3, Lemma 1.11] there exists a sequence $\alpha \in \mathfrak{Q}$ and $\lim x\alpha \in K_1$ for all $x \in E(X)$. Consider the syndetic subgroup $A = \{t \in T | xt = x \text{ for all } x \in N(X)\}$ and the transformation group (X,A). By Lemma 5 the set of nonequicontinuous points remains unchanged. Note that under A the set of $N(X)$ consists of fixed points. Now since $E(X)$ is dense in X by a similar argument as given in the proof of Theorem 3, we see that if K_2 and K_3 are two other minimal preferred sets contained in $N(X)$, then K_2 and K_3 consist of nonequicontinuous points of α. The same argument applies to show that $K_2 = K_3$. Hence $E(X)$ has at most 3 minimal preferred sets and the argument is completed.

REFERENCES

1. Robert Ellis, *Lectures on topological dynamics*, Benjamin, New York, 1969. MR **42** #2463.

2. W. H. Gottschalk and G. A. Hedlund, *Topological dynamics*, Amer. Math. Soc. Colloq. Publ., vol. 36, Amer. Math. Soc., Providence, R.I., 1955. MR **17**, 650.

3. Ping-Fun Lam, *Equicontinuity and indivisibility in transformation groups*, Trans. Amer. Math. Soc. **174** (1972), 399–424. MR **46** #8201.

4. ______, *Almost equicontinuous transformation groups*, Trans. Amer. Math. Soc. **195** (1974), 165–200. MR **51** #14126.

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