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CONDITIONS FOR RESTRICTED TRANSLATION OPERATORS TO BELONG TO S_p

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ABSTRACT. Let B be a Blaschke product, let P be the orthogonal projection of $L^2(R^1)$ onto the left invariant subspace $H^2 \ominus BH^2$ and let E_t: $f(\gamma) \to e^{-i\gamma t}f(\gamma)$. Conditions are given on the roots of B for PE_tP to belong to S_p, $0 < p < \infty$.

1. **Introduction.** This paper gives conditions for membership in S_p, $0 < p < \infty$, for the operators $K_t = PE_tP$, $t > 0$, in which E_t designates multiplication by $e^{-i\gamma t}$: $E_tf(\gamma) = e^{-i\gamma t}f(\gamma)$, and P stands for the orthogonal projection of $L^2(R^1)$ onto the "left invariant" subspace $H^2 \ominus BH^2$. Here H^2 is the Hardy space of class 2 over the upper-half plane, and

$$B(\gamma) = \prod_{k=1}^{\infty} \left(\frac{1 - \gamma/\omega_k}{1 - \gamma/\omega_k^*} \right)$$

is the Blaschke product with *infinitely* many roots [poles]

$$\omega_k = a_k + ib_k \quad [\omega_k^* = a_k - ib_k], \qquad b_k > 0,$$

which are subject to the usual constraint

$$\sum_{k=1}^{\infty} \frac{b_k}{|\omega_k|^2 + 1} < \infty.$$

Recall that K_t belongs to S_p, $0 < p \leqslant \infty$, if (1) it is compact, and (2) its s numbers belong to l_p. The s numbers, $s_1(t), s_2(t), \ldots,$ of K_t are the positive square roots of the eigenvalues of $K_t^* K_t$ arranged in decreasing size: $s_1(t) \geqslant s_2(t) \geqslant \ldots$. Gohberg and Krein [2] is recommended for additional information on these spaces.

Let

$$\alpha = \alpha_B = \limsup_{|a| \to \infty} \sum_{k=1}^{\infty} \frac{2b_k}{|\omega_k - a|^2}, \qquad a \in R^1,$$

and let

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$$\beta = \beta_B = \text{the abscissa of convergence of } \sum_{n=1}^{\infty} e^{-b_n t}.$$

AMPLIFICATION. If $b_1 \leqslant b_2 \leqslant \ldots$, then

$$\beta = \limsup_{n\uparrow\infty} \frac{\log n}{b_n};$$

see p. 45 of Widder [7].

The chief conclusions of this paper are:

THEOREM 2. *In order for K_t to belong to S_p, $0 < p < \infty$, it is sufficient that $t > [\sqrt{(\beta/p)} + \sqrt{\alpha}\,]^2$, and it is necessary that $t \geqslant \max[(\beta/p),\alpha]$.*

THEOREM 3. *K_t belongs to S_p, $0 < p < \infty$, for every $t > 0$ if and only if $\alpha = \beta = 0$.*

Theorem 3 is an immediate corollary of Theorem 2. The condition $\beta = 0$ means that b_n must tend to ∞ rapidly enough as $n\uparrow\infty$ so that $\sum_{n=1}^{\infty} e^{-b_n t} < \infty$ for every $t > 0$. It is interesting to compare the conclusions of Theorem 3 with those of Koosis [3] and Lax [4] who proved that *K_t is compact for every $t > 0$ if and only if $\alpha = 0$ and $\lim_{n\uparrow\infty} b_n = \infty$.* The proof of Theorem 2 is given in §4. It depends in part upon the following refinement of the Koosis-Lax conditions for compactness.

THEOREM 1. *In order for K_t to be compact for a fixed $t > 0$, it is necessary that $t \geqslant \alpha$ and $\lim_{n\uparrow\infty} b_n = \infty$, and it is sufficient that $t > \alpha$ and $\lim_{n\uparrow\infty} b_n = \infty$.*

This refinement must have been known to Moeller and Frederickson since the estimates used in their paper [6] can be adapted to prove Theorem 1. Nevertheless, since the result is not stated explicitly there, and the connections between their work and ours are not altogether transparent, a proof seems to be called for. The details will be given in §3.

Finally, it should be noted that the space $H^2 \ominus BH^2$ is the image under the Fourier transform of the closed linear span in $L^2(0,\infty)$ of the functions

$$e^{-i\omega_k^* x}, \qquad k = 1,2\ldots,$$

augmented by

$$xe^{-i\omega_k^* x}, x^2 e^{-i\omega_k^* x}, \ldots, x^{m-1} e^{-i\omega_k^* x},$$

in case that ω_k is a root of order m of the Blaschke product B; see Lax [4] for help with this. In this formulation K_t corresponds to left translation in the amount t, and the fact that *K_t is a semigroup of bounded operators*: $K_{t+s} = K_t K_s$ becomes transparent. The semigroup property implies that *K_{t+s} belongs to S_p, $0 < p \leqslant \infty$, for every $s \geqslant 0$ as soon as K_t does.*

2. **Preliminaries.** In this section a number of facts and elementary estimates are prepared for future use.

NOTATION. H^1, H^2, H^∞, denote the Hardy spaces of classes 1,2 and ∞, respectively, over the upper half-plane, $\omega^* = a - ib$ denotes the complex conjugate of $\omega = a + ib$, and the limits of integration in all unmarked integrals such as $\int f(\gamma)\, d\gamma$ are $\pm\infty$. The orthogonal projection of $L^2(R^1)$ onto H^2 $[H^2 \ominus BH^2]$ is designated by $\mathfrak{p}$ $[P]$.

LEMMA 1. *$K_t = \mathfrak{p}E_tP$ and $K_{t+s} = K_tK_s$ for $s,t \geqslant 0$.*

PROOF. An easy computation shows that $\mathfrak{p}E_tP$ maps $H^2 \ominus BH^2$ into itself. Therefore $K_t = PE_tP = P\mathfrak{p}E_tP = \mathfrak{p}E_tP$. This justifies the first assertion. The second is self-evident from the remarks in the introduction. For a direct proof in the language of K_t see Dym [1].

LEMMA 2. *The set of functions*

$$\phi_n(\gamma) = \sqrt{\frac{b_n}{\pi}} \prod_{k=n}^{\infty} \frac{1-\gamma/\omega_k}{1-\gamma/\omega_k^*} \Big/ (\gamma - \omega_n), \qquad n = 1,2,\ldots,$$

is an orthonormal basis for $H^2 \ominus BH^2$.

PROOF. The orthonormality is an easy consequence of Cauchy's formula. The completeness proof is slightly more involved and rests upon the observation (which the reader may verify directly) that if $f \in H^2 \ominus BH^2$, then $f = Bg^*$ with $g \in H^2 \ominus BH^2$ (see Dym [1] for details). Therefore,

$$\int \phi_n(\gamma)[f(\gamma)]^*\, d\gamma = \sqrt{\frac{b_n}{\pi}} \int \left(\prod_{k=1}^{n-1} \frac{1-\gamma/\omega_k^*}{1-\gamma/\omega_k} \Big/ (\gamma - \omega_n) \right) g(\gamma)\, d\gamma$$

$$= \sum_{k=1}^{n} c_k g(\omega_k) \qquad (c_k \neq 0),$$

assuming, for the sake of a simpler exposition, that the roots of B are distinct; otherwise some derivatives will intervene. But this makes it plain that if f is orthogonal to all the ϕ_n, $n = 1,2,\ldots$, then g must vanish at all points ω_n, $n = 1,2,\ldots$. This exhibits g as an element of both BH^2 and $H^2 \ominus BH^2$. Hence $g = 0$ and $f = Bg^* = 0$.

LEMMA 3. *Let P_n denote the orthogonal projection onto the closed linear span of ϕ_k, $k \geqslant n$, and let*

$$B_n(\gamma) = \prod_{k=n}^{\infty} \left(\frac{1-\gamma/\omega_k}{1-\gamma/\omega_k^*} \right).$$

Then

$$\|K_t P_n\| \leqslant e^{-bt} \Big/ \inf_a |B_n(a + ib)|$$

for any choice of $b < \inf\{b_k\colon k \geqslant n\}$.

PROOF. This bound is essentially due to Koosis [3] and may be deduced from the formula

$$\|K_t P_n\| = \sup\left|\int \frac{e^{i\gamma t}}{B_n(\gamma)} f(\gamma)\, d\gamma\right|, \qquad f \in H^1, \|f\|_1 = 1,$$

for the norm of $K_t P_n = \mathfrak{p} E_t P P_n = \mathfrak{p} E_t P_n = P_n E_t P_n$ by displacing the line of integration into the upper half-plane. The formula for the norm is implicit in the work of Koosis [3]; an explicit proof may also be found in Dym [1]. The stated identification of $K_t P_n$ follows from Lemmas 1 and 2.

LEMMA 4. *The bounds*

$$\exp\left\{ \frac{2bv}{|\omega - a|^2}\left[1 - \frac{\delta^2}{1 - \delta^2}\right]\right\} \leqslant \left|\frac{1 - ib/(\omega^* - a)}{1 - ib/(\omega - a)}\right|$$

$$\leqslant \exp\left\{ \frac{2bv}{|\omega - a|^2}\left[1 + \frac{\delta^2}{1 - \delta^2}\right]\right\}$$

prevail for every pair of points $a + ib$, *and* $\omega = u + iv$ *in the open upper half-plane with* $b/v \leqslant \delta < 1$.

PROOF. This is a refinement of an estimate due to Koosis [3] and is similar in content to Lemma 4 of [6]. The details can be carried out much as in the proof of Lemma 3 of Dym [1]. Indeed you have only to set $\xi = ib$ and $\zeta = \omega - a$ in that paper, and to check that the real part of $\alpha|\xi/\zeta|$ is bounded by $\delta^2(1 - \delta^2)^{-1}$ since only the summands with odd index in the sum on the top of p. 396 in [1] intervene.

LEMMA 5. *If* $\lim_{k\uparrow\infty} b_k = \infty$, *then for any* $\varepsilon > 0$ *and any* $0 < \delta < 1$, *there exists an* n_0 *such that* $\|K_t P_n\| \leqslant \exp\{-\delta b_n[t - (\alpha + \varepsilon)(1 - \delta^2)^{-1}]\}$ *for every* $n \geqslant n_0$.

PROOF. Given ε and δ choose n_0 so large that $b \leqslant \delta b_k$ for $k \geqslant n_0$ and $\Sigma_{k=n}^{\infty} 2b_k/|\omega_k - a|^2 \leqslant \alpha + \varepsilon$ for $n \geqslant n_0$. Then, by Lemma 4,

$$|B_n(a + ib)|^{-1} \leqslant \exp\left\{ \sum_{k=n}^{\infty} \frac{2bb_k}{|\omega_k - a|^2}(1 - \delta^2)^{-1}\right\}$$

$$\leqslant \exp\left\{ b(\alpha + \varepsilon)(1 - \delta^2)^{-1}\right\}$$

for $n \geqslant n_0$. To finish, invoke Lemma 3 with $b = \delta b_n$.

LEMMA 6. *The eigenvalues* $\lambda_k(t)$, $k = 1,2, \ldots$, *of* K_t *are in* 1:1 *correspondence with the roots* ω_k, $k = 1,2, \ldots$, *of* B:

$$\lambda_k(t) = e^{-i\omega_k^* t}.$$

The corresponding system of root vectors is complete.

PROOF. In the case of distinct roots it is easily checked that $K_t\psi_k = \lambda_k\psi_k$ with $\lambda_k = \lambda_k(t) = \exp(-i\omega_k^* t)$ and $\psi_k(\gamma) = (\gamma - \omega_k^*)^{-1}$.

If, say, ω_1 has multiplicity p, i.e., $\omega_1 = \omega_2 = \cdots = \omega_p$, then the corresponding root vectors are given by

$$\psi_k(\gamma) = (\gamma - \omega_1^*)^{-k}, \qquad k = 1, \ldots, p,$$

and a routine calculation, based upon the fact that

$$\begin{aligned}(K_t\psi_k)(\omega) &= \int \frac{e^{-i\gamma t}}{\gamma - \omega}\left(\frac{1}{\gamma - \omega_1^*}\right)^k \frac{d\gamma}{2\pi i} \\ &= -\frac{1}{(k-1)!}\frac{d^{k-1}}{d\gamma^{k-1}}\left[\frac{e^{i\gamma t}}{\gamma - \omega}\right]\Bigg|_{\gamma = \omega_1^*}\end{aligned}$$

for points ω in the open upper half-plane, leads to the identity

$$K_t\psi_k = e^{-i\omega_1^* t}\left[\sum_{j=1}^{k} \frac{(-it)^{k-j}}{(k-j)!}\psi_j\right].$$

This exhibits $\psi_1, \ldots, \psi_p$ as root vectors associated with the eigenvalue $\exp(-i\omega_1^* t)$. The full set of root vectors is complete in $H^2 \ominus BH^2$ because any f in that space which is orthogonal to all of them belongs to both BH^2 and $H^2 \ominus BH^2$ and so must vanish.

3. **Conditions for compactness.** The next two lemmas are due to Koosis [3]. For the sake of completeness and the convenience of the reader, new and somewhat quicker proofs will be given.

NOTATION. (f,g) denotes the standard inner product $\int fg^*$ in L^2.

LEMMA 7. *Let* P_n *denote the orthogonal projection introduced in Lemma* 3. *Then* K_t *is compact if and only if* $\|K_tP_n\| \to 0$ *as* $n\uparrow\infty$.

PROOF. If $\|K_tP_n\| \to 0$, as $n\uparrow\infty$, then $K_t = K_t(I - P_n) + K_tP_n$ can be approximated in norm by the finite dimensional operator

$$K_t(I - P_n): f \to \sum_{k=1}^{n-1} (f,\phi_k)K_t\phi_k,$$

and so K_t must be compact. Conversely, if K_t is compact, then for every choice of n, $n = 1,2, \ldots$, there exists an element u_n in $P_n(H^2 \ominus BH^2)$ with $\|u_n\| = 1$ such that

$$\|K_tP_n\| = \|K_tP_nu_n\| = \|K_tu_n\|.$$

Moreover, the bound

$$|(u_n,g)|^2 = |(u_n,P_ng)|^2 \leqslant \|P_ng\|^2 = \sum_{k=n}^{\infty} |(g,\phi_k)|^2,$$

which is valid for every $g \in L^2$, implies that u_n tends weakly to zero in L^2 as $n\uparrow\infty$. Hence,

$$\lim_{n\uparrow\infty} \|K_tP_n\| = \lim_{n\uparrow\infty} \|K_tu_n\| = 0,$$

since K_t is compact, and the proof is complete.

LEMMA 8. *If K_t is compact then* $\lim_{k\uparrow\infty} b_k = \infty$.

PROOF. This is immediate from Lemma 6 since the eigenvalues of an infinite dimensional compact operator must tend to 0.

PROOF OF THEOREM 1. The sufficiency is immediate from Lemma 7 and the bound of Lemma 5: You have only to choose ε and δ in the latter so as to be positive and yet small enough so that $0 < (\alpha + \varepsilon)(1 - \delta^2)^{-1} < t$. The proof of necessity is adapted from Moeller and Frederickson [6]. A basic ingredient is an observation credited to Lax [4], [5] (although he credits it to Müntz and Szász), which in our notation amounts to the identity

$$|B_n(a + ib)| = \|(I - P_n)\varphi\|_2$$

for $\varphi = \varphi(\gamma) = (b/\pi)^{1/2}(\gamma - \omega^*)^{-1}$ and $\omega = a + ib$ with $b > 0$. Because of it,

$$\begin{aligned} e^{-bt} &= \|e^{-i\omega^* t}\varphi\|_2 = \|\mathfrak{p}E_t\varphi\|_2 \\ &= \|K_tP_n\varphi + \mathfrak{p}E_t(I - P_n)\varphi\|_2 \\ &\leqslant \|K_tP_n\| + |B_n(a + ib)| \\ &\leqslant \|K_tP_n\| + \exp\left\{ -\sum_{k=n}^{\infty} \frac{2b_kb}{|\omega_k - a|^2}\left(1 - \frac{\delta^2}{1-\delta^2}\right)\right\} \end{aligned}$$

presuming, as you may in view of Lemma 8, that $|b/b_k| \leqslant \delta < \frac{1}{2}$ for $k \geqslant n$ and n large enough. But this implies that

$$e^{-bt} \leqslant \|K_tP_n\| + \exp\left\{ -\alpha b(1 - \delta^2)^{-1}(1 - 2\delta^2)\right\},$$

and so, because of Lemma 7, you have only to let $n\uparrow\infty$, and $\delta\downarrow 0$, in that order, to complete the proof.

4. Proof of Theorem 2.

PROOF OF SUFFICIENCY. The assumptions imply, in particular, that $t > \alpha$ and that $\lim_{n\uparrow\infty} b_n = \infty$. Therefore K_t is compact, by Theorem 1, and you may assume that the roots are indexed so as to have $b_1 \leqslant b_2 \leqslant \dots$. Now the minimax characterization of eigenvalues implies that the nth s number of K_t is subject to the bound $s_n(t) \leqslant \|K_t P_n\|$. Therefore, by Lemma 5, you see that for any $\varepsilon > 0$ and $0 < \delta < 1$ there exists an n_0 such that

$$\left[s_n(t)\right]^p \leqslant \exp\left\{-p\delta b_n\left[t - (\alpha + \varepsilon)(1 - \delta^2)^{-1}\right]\right\}$$

for all $n \geqslant n_0$. Consequently, $\sum_{n=1}^{\infty} [s_n(t)]^p$ will converge for

$$t > \limsup_{n\uparrow\infty} \frac{\log\left|\sum_{k=1}^{n} \exp\left\{p\delta b_k(\alpha + \varepsilon)(1 - \delta^2)^{-1}\right\}\right|}{p b_n \delta}$$

as follows from the formula for the abscissa of convergence for Dirichlet series; see p. 45 of Widder [7]. But now the right-hand side is

$$\leqslant \limsup_{n\uparrow\infty} \frac{\left[\log n + p\delta b_n(\alpha + \varepsilon)(1 - \delta^2)^{-1}\right]}{p b_n \delta}$$

$$\leqslant \frac{\beta}{p\delta} + \frac{(\alpha + \varepsilon)}{1 - \delta^2} \leqslant \frac{\beta}{p\delta} + \frac{\alpha + \varepsilon}{1 - \delta};$$

and so in order for K_t to belong to S_p it suffices to have

$$t > \beta/p\delta + \alpha/(1 - \delta)$$

for some choice of $0 < \delta < 1$. The final condition is attained by choosing δ so as to minimize the right-hand side.

PROOF OF NECESSITY. To begin with, K_t must be compact and so, by Theorem 1, you must have $t \geqslant \alpha$. Moreover, by Lemma 6 and p. 41 of Gohberg and Krein [2], it is also necessary that

$$\sum_{n=1}^{\infty} e^{-b_n t p} = \sum_{n=1}^{\infty} |\lambda_n(t)|^p < \infty$$

and, hence, that $pt \geqslant \beta$. The proof is complete.

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