ON ANOSOV DIFFEOMORPHISMS ON THE PLANE

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ABSTRACT. A diffeomorphism on the plane holding an integrable Anosov splitting is called an Anosov diffeomorphism on the plane. In this paper the author proves that Anosov diffeomorphisms on the plane are structurally stable.

We start with some notations and definitions.

R^2 and $\mathrm{Diff}(R^2)$ stand, respectively, for the plane and its C^1-diffeomorphisms with the C^1-Whitney topology. Let $\mathrm{Per}(f)$ be the set of periodic points of f and $\Omega(f)$ the set of its nonwandering points. If $x \in R^2$, $\alpha_f(x)$ and $\omega_f(x)$ denote the α-limit set and the ω-limit set of x.

$f, g \in \mathrm{Diff}(R^2)$ are conjugated if there is an homeomorphism $h\colon R^2 \to R^2$ satisfying the equation $gh = hf$.

$f \in \mathrm{Diff}(R^2)$ is structurally stable if it is conjugated with every g in some neighborhood of f.

An Anosov splitting for $f \in \mathrm{Diff}(R^2)$ consists of a complete Riemannian metric μ on TR^2, a continuous invariant splitting $TR^2 = E^s \oplus E^u$, and a constant $\lambda \in (0, 1)$, such that

$$\|Df^n(v)\| \leqslant \lambda^n \|v\|, \quad \text{for } v \in E^s$$

$$\|Df^n(v)\| \leqslant \lambda^{-n} \|v\|, \quad \text{for } v \in E^u,$$

where $E^s(x)$ and $E^u(x)$ has dimension 1 for all x, and $\| \ \|$ is the norm induced by μ. By the contractibility of the plane there are unit continuous vector fields v^s and v^u, on R^2, generating E^s and E^u, respectively. A diffeomorphism $f\colon R^2 \to R^2$ admitting an Anosov splitting will be called an Anosov diffeomorphism if v^s and v^u have unique integral curves through each point of R^2, that is, if E^s and E^u are integrable line fields on the plane.

Let F^s and F^u be the arising continuous foliations. Set $L_\infty = L_\infty(f) = \{x \in R^2\colon \alpha_f(x) = \phi \text{ or } \omega_f(x) = \phi\}$. We recall that ∞ is f-strong attractor of $x \in L_\infty$ if there is a neighborhood V_x of x such that each compact set $K \subset R^2$ intersects only a finite number of the sets $f^n(V_x)$, $n \geqslant 0$. Dually, we define ∞ is f-strong repeller of $x \in L_\infty$. Our result is the following

THEOREM 1. *The Anosov diffeomorphisms of the plane are structurally stable.*

We recall a structural stability theorem of [2].

THEOREM. *Let* $f \in \mathrm{Diff}(R^2)$ *be such that*
(1) $\Omega(f)$ *is discrete*;
(2) *the periodic points of* f *are hyperbolic*;
(3) *the invariant manifolds of its periodic points are in general position*;
(4) ∞ *is* f-*strong attractor or* f-*strong repeller of each point in* L_∞.
Then f *is structurally stable.*

The proof of Theorem 1 is done using this theorem and the following propositions.

PROPOSITION 1.1. *If* $f \in \mathrm{Diff}(R^2)$ *has an Anosov splitting,* $\Omega(f) = \mathrm{Per}(f)$.

PROPOSITION 1.2. *If* $f \in \mathrm{Diff}(R^2)$ *is Anosov,* $\Omega(f) = \phi$ *or* $\Omega(f) = \{p\}$.

PROPOSITION 1.3. *If* $f \in \mathrm{Diff}(R^2)$ *is Anosov and* $x \in L_\infty$ *then* ∞ *is* f-*strong attractor or* f-*strong repeller of* x.

Trivial examples of Anosov diffeomorphisms on the plane are the linear mappings with real eigenvalues λ_1, λ_2 such that $0 < |\lambda_1| < 1$ and $|\lambda_2| > 1$. The translations are the only nontrivial examples up to the present, [5].

Conjecture. Each Anosov diffeomorphism on the plane is conjugate to one of the examples above.

Finally we state two problems.

Problem 1. Is the set of Anosov diffeomorphisms on the plane open in $\mathrm{Diff}(R^2)$?

Problem 2. If $f \in \mathrm{Diff}(R^2)$ has an Anosov decomposition is f Anosov?

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PROOF OF PROPOSITION 1.1. Let $\alpha \colon \mathbf{R} \to \mathbf{R}^2$ be an orbit of v^s and $t_n \in \mathbf{R}$ be a sequence such that $t_n \to \infty$ and $\alpha(t_n) \to p \in \mathbf{R}^2$. Then for some large n there is an arc l from $\alpha(t_n)$ to $\alpha(t_{n+1})$ such that the closed piecewise differentiable curve $\Gamma = l \cup \alpha([t_n, t_{n+1}])$ is transversal to v^u. To see the existence of l consider a small box

$$B = \{ p + x_s V^s(p) + x_u V^u(p), x_s, x_u \in (-\varepsilon, \varepsilon) \subset \mathbf{R} \}$$

such that the angle between the vectors $V^s(x)$ and $V^s(p)$ is less than the one between the vectors $V^u(p)$ and $V^s(p)$, for all $x \in B$, and take n such that $\alpha(t_n), \alpha(t_{n+1}) \in B$. The existence of Γ contradicts the fact that V^u is a continuous unit vector field. This argument shows that α is an embedding with closed unbounded image separating the plane. This is just the case for the stable manifold $W^s(p)$ of $p \in \mathrm{Per}(f)$. Setting $W^s = \bigcup_{p \in \mathrm{Per}(f)} W^s(p)$, from the unboundedness of each $W^s(p)$ it follows that the connected components $U_i, i = 1, 2, \ldots$ of $R^2 - \overline{W^s}$ are simply connected open sets and thus diffeomorphic to R^2, by the Riemann mapping theorem, [1].

The set $R^2 - \overline{W^s}$ is f-invariant and then, f^n-invariant for all $n \in Z$. For

each n, f^n sends connected components of $\mathbf{R}^2 - \overline{W^s}$ onto connected components of $\mathbf{R}^2 - \overline{W^s}$, that is, $f^n(U_i) = U_j$ for some j. Fix U_i. We have either, $f^{n_i}(U_i) = U_i$ for some n_i, or $f^n(U_i)$, $n \in Z$, are disjoint open sets. In the first case the restriction $f^{n_i}_{|U_i}: U_i \cong R^2 \hookleftarrow$ is a diffeomorphism of R^2 with no periodic points. Then, from [2], we have $\Omega(f^{n_i}_{|U_i}) = \phi$, which implies $\Omega(f) \cap U_i = \phi$. In the second, of course we have also $\Omega(f) \cap U_i = \phi$. Thus $\Omega(f) \subset \overline{W^s}$. Similarly

$$\Omega(f) \subset \overline{W^u} = \overline{\bigcup_{p \in \mathrm{Per}(f)} W^u(p)}$$

and then $\Omega(f) \subset \overline{W^s} \cap \overline{W^u}$. Using the linear independence of v^s and v^u and the λ-Lemma [3], we get $\overline{W^s} \cap \overline{W^u} = \mathrm{Per}(f)$, which ends the proof.

PROOF OF PROPOSITION 1.2. We note that (1.2) being true for some power of f it will be true for f itself. Then we have only to prove the following:

(∗)   If $f \in \mathrm{Diff}(R^2)$ is Anosov and $p \in \Omega(f)$ is a fixed point then $\Omega(f) = \{p\}$.

To prove (∗) we will use some common expressions as "on the right", "on the left", "above", "below", etc., with the obvious technical meaning in each case. Let $x_1, x_2 \in W^s(p)$, x_1 on the right of p and x_2 on its left, and $y_1, y_2 \in W^u(p)$, y_1 above p, and y_2 below. Let α, β be the orbits of v^u through x_1, x_2, and γ, δ the orbits of v^s through y_1, y_2, respectively. If x_i, y_i, $i = 1, 2$, are sufficiently near p, $\alpha, \beta, \gamma, \delta$ bound a square Q containing p with the following property: if $q \in \mathrm{Per}(f) \cap Q$, $q \neq p$, then the orbit of q has points outside Q.

Then it is enough to prove that f has no periodic points outside Q, (1) on the right of α; (2) on the left of β; (3) above γ; and (4) below δ. Now we proceed to prove (1).

Let B be the band bounded by α and $f(\alpha)$. It is clear that $f^n(B)$ is the band bounded by $f^n(\alpha)$ and $f^{n+1}(\alpha)$, $n \in Z$, and then $\mathrm{int}\, f^n(B) \cap \mathrm{int}\, f^m(B) = \phi$ if $n \neq m$. Thus $\mathrm{Per}(f) \cap (\bigcup_{n \in Z} f^n(B)) = \phi$. We note that $\bigcup_{n \in Z} f^n(B)$ is open. Moreover,

(∗∗)   if $x \in R^2$ is on the right of α and on the left of $f^{-n}(\alpha)$, $n > 0$, then $x \in \bigcup_{n > 0} f^{-n}(B)$.

Let P be the open region on the right of α. Set $A^- = \bigcup_{n \in Z} f^n(B) \cap P$, $A^+ = P - \overline{A}$ and $F = \sigma A^- = \sigma A^+$ (σ = topological boundary). From what was stated above we get $f(F) = F$ and $f(A^+) = A^+$. Consider a connected component C of A^+, $x \in \sigma C$ and α_x the orbit of v^u through x. The orbit α_x separates P in two open regions P^+ and P^-, with P^+ on the right of α_x. Using (∗∗) we get by contradiction that $P^+ \cap \overline{A^-} = \phi$. Then $P^+ \subset C$. This

and the local triviality of the continuous foliation F^u, assure that $P^+ = C$, and that α_x is a connected component of F.

If $\text{Per}(f) \cap A^+ \neq \phi$, $f^n(C) = C$ for some connected component of A^+ and some $n > 0$, just because $f(A^+) = A^+$. Thus $f^n(\alpha_x) = \alpha_x$ for the connected component $\alpha_x = \sigma C$ of F. α_x being an orbit of v^u, $f^{-n}|_{\alpha_x}: \alpha_x \to \alpha_x$ is a contraction and then has a fixed point q.

It is easy to check that $f^{-m}(\alpha_x) \cap W^s_{f^n}(q) = \{y\}$. We note that q is on the right of $f^{-m}(\alpha)$, and then $f^n(y)$ is also on the right of $f^{-m}(\alpha)$. Thus the same occurs with $f^n(f^{-m}(\alpha))$ which is a contradiction. This proves that $\text{Per}(f) \cap A^+ = \phi$. In the same way we conclude that $\text{Per}(f) \cap F = \phi$, which ends the proof of (1). The proofs of (2), (3) and (4) are similar, and (1.2) is proved.

Up to now we have been analysing the nonwandering set of an Anosov diffeomorphism concluding that it is very nice: it is empty or consists of a single point. Now we analyse the set L_∞.

PROOF OF PROPOSITION 1.3. Take a neighborhood Q of x trivial for the continuous foliations F^s and F^u, such that $f^n(Q) \cap Q = \phi$ for all $n \neq 0$. Let $\beta^s = \gamma^s_x \cap Q$ and $\beta^u = \gamma^u_x \cap Q$, where $x \in \gamma^s_x \in F^s$ and $x \in \gamma^u_x \in F^u$. They can be supposed connected by taking Q sufficiently small. If ∞ is not f-strong attractor of x there are points $y_i \in Q$, $i = 1, 2, \ldots$, and integers $0 < n_1 < n_2 < \ldots$ such that $f^{n_i}(y_i) \to y$, for some $y \in R^2$.

Consider the points $\tilde{y}_i = \beta^u \cap \gamma^s_y$ and the arcs $\alpha_i \subset \gamma^s_{y_i}$, from y_i to $\tilde{y}_i$, $i = 1$, $2, \ldots$. Let $l(\alpha_i)$ be the length of α_i. It is easy to see that there is $a = \sup_i l(\alpha_i)$, and that $l(f^{n_i}(\alpha_i)) \leqslant a\lambda^{n_i}$. Thus $d(f^{n_i}(y_i), f^{n_i}(\tilde{y}_i)) \to 0$ and then $f^{n_i}(\tilde{y}_i) \to y$. From this we get i_0 such that $f^{n_i}(\beta^u) \cap \gamma^s_y \neq \phi$ for all $i \geqslant i_0$, by taking a neighborhood of y trivial for both foliations, F^s and F^u. Setting $f^{-n_{i_0}+1}(\gamma^s_y) = \eta$ and $r = n_{i_0+1} - n_{i_0}$ we have $\eta \cap \beta^u \neq \phi$ and $f^r(\eta) \cap \beta^u \neq \phi$.

It is clear from the definition that if ∞ is f^r-strong repeller of x, ∞ is also f-strong repeller of x. Supposing, by contradiction that ∞ is not f-strong repeller of x, and thus, it is not f^r-strong repeller of x, we get by the arguments above an orbit ξ of v^u and an integer $n > 0$, such that $\xi \cap \beta^s \neq \phi$ and $(f^r)^n(\xi) \cap \beta^s \neq \phi$. It is enough to prove that $f^r(\xi) \cap \beta^s \neq \phi$ because this implies that $f^r(Q) \cap Q \neq \phi$, which is a contradiction.

This will be proved by contradiction again. If $f^r(\xi) \cap \beta^s = \phi$ we have $f^r(\xi) \cap \xi = \phi$. Noting that ξ separates R^2 in two open regions, let P be the closure of the one containing $f^r(\xi)$. We can suppose without loss of generality that f^r is orientation preserving. Thus

(a) $f^r(P) \subset \text{int } P$,

(b) $f^r(P) \cap \beta^s = \phi$.

From (a) it follows by induction that $f^{nr}(P) \subset \text{int } f^r(P)$. Then from (b) we get $f^{nr}(\xi) \cap \beta^s = \phi$ which is a contradiction.

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