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EXTENSION OF INVARIANT LINEAR FUNCTIONALS: A SEQUEL TO FAN'S PAPER

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ABSTRACT. We establish the relationship between certain invariant extension properties for linear functionals of K. Fan when a semigroup S acts on a locally convex topological vector space and the existence of a left-invariant mean on the space of almost periodic and weakly almost periodic functions on S.

1. **Introduction.** Recently Fan [1] (see also [2]) showed that when a group S acts on a real Hausdorff locally convex topological vector space E such that for each $x \in E$ the orbit $S(x) = \{s \cdot x, s \in S\}$ is (weakly) relatively compact, then it has certain extension properties for invariant linear functionals. He also pointed out in [1] and [2] that Theorem 3 in [2] (and therefore Theorem 1 in [1]) implies von Neumann's mean value theorem for almost periodic functions. It is the purpose of the present note to clarify more fully this relationship between the extension properties of S for invariant linear functionals and the existence of a left-invariant mean on the space of (weakly) almost periodic functions on S. Our approach provides a unified and different proof to both Theorems 1 and 3 in [1]. It also shows that the completeness hypothesis in Theorem 3 of [1] can be removed.

It is our pleasure to thank Professor Ky Fan for providing us with a preprint of his work [1].

Throughout this paper, S will denote a topological semigroup. By a *continuous* (*resp. weakly continuous*) *right action* of S on a topological vector space $(E, \mathfrak{T})$ we shall mean a map $S \times E \to E$ denoted by $(s, x) \to s \cdot x$ such that

(1) $(ab) \cdot x = b \cdot (a \cdot x)$ for each $a, b \in S$ and $x \in X$.

(2) For each $s \in S$, the map $x \to s \cdot x$ is a continuous linear map from $(E, \mathfrak{T})$ into $(E, \mathfrak{T})$.

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(3) For each $x \in X$, the map $s \to s \cdot x$ is continuous from S into $(E, \mathfrak{T})$ (resp. $(E$, weak$)$).

The action of S on E is *almost periodic* (*resp. weakly almost periodic*) if for each $x \in E$, the orbit $S(x)$ is relatively compact in the $\mathfrak{T}$ topology (resp. weak topology).

Let $C(S)$ denote the Banach space of bounded continuous real-valued functions on S with the supremum norm. For each $a \in S, f \in C(S)$, let $(l_a f)(s) = f(as)$ for each $s \in S$. Let $AP(S)$ (resp. $WAP(S)$) denote the closed subspace of $C(S)$ consisting of all $f \in C(S)$ such that $\{l_a f; a \in S\}$ is relatively compact in the norm topology (resp. weak topology) of $C(S)$. A linear functional ϕ on $AP(S)$ (or $WAP(S)$) is a *left-invariant mean* if $\|\phi\| = \phi(1) = 1$ and $l_a^* \phi = \phi$ for each $a \in S$, where l_a^* denotes the transpose of l_a.

2. The mean theorems.

THEOREM 1. *The following conditions on a topological semigroup S are equivalent*:

(a) *$AP(S)$ has a left-invariant mean.*

(b) *For any almost periodic continuous right action of S on a Hausdorff real topological vector space E, if L is an invariant subspace of E and K is a convex subset of E such that $K - x_0$ is invariant for some x_0 contained in $L \cap \operatorname{Int} K$, then for each invariant linear functional ϕ on L such that $\phi(x) \leqslant \alpha$ for all $x \in L \cap K$ and some fixed real number α, then there exists an invariant linear extension $\tilde{\phi}$ of ϕ to E such that $\tilde{\phi}(x) \leqslant \alpha$ for all $x \in K$.*

PROOF. (a) *implies* (b). We may assume that ϕ is nonzero. In this case $\phi(x_0) < \alpha$. As in the proof of Theorem 2 [1], define ϕ_0 on L:

$$\phi_0(x) = \phi(x)/(\alpha - \phi(x_0))$$

and let $K_0 = K - x_0$. Then $\phi_0(x) \leqslant 1$ for all $x \in L \cap K_0$. Let p be the Minkowski functional on E defined by K_0 i.e. $p(x) = \inf\{\lambda > 0; x \in \lambda K_0\}$. Then p is sublinear, $p(s \cdot x) \leqslant p(x)$ for all $s \in S$ and $x \in E$, and $\phi_0 \leqslant p|L$. By the Hahn Banach extension theorem, ϕ_0 has an extension η to E such that $\eta \leqslant p$. For each $x \in E$, define $(T_x \eta)(s) = \eta(s \cdot x)$. Then clearly $T_x \eta \in C(S)$. Observe that if $x, y \in E$ and $z = x - y$, then $T_x \eta(s) - T_y \eta(s) = \eta(s \cdot z) \leqslant p(s \cdot z) \leqslant p(z)$ for each $s \in S$. Hence $\|T_x \eta - T_y \eta\| \leqslant \max\{|p(z)|, |p(-z)|\}$. Consequently the map $x \to T_x \eta$ is continuous from E into $C(S)$ with the sup norm topology. Also since $l_a(T_x \eta) = T_{a \cdot x} \eta$, it follows that $T_x \eta$ is almost periodic.

Let m be a left-invariant mean on $AP(S)$. Define $\psi_0(x) = m(T_x \eta)$. Then ψ_0 is an invariant extension of ϕ_0. For $x \in L$ and $s \in S$, we have $(T_x \eta)(s)$

$= \eta(s \cdot x) = \phi_0(s \cdot x) = \phi_0(x)$. So for each $x \in L$, $T_x\eta \in C(S)$ is the constant function $\phi_0(x)$, and therefore $\psi_0(x) = \phi_0(x)$. Also $\psi_0 \leqslant p$. Consequently $\psi_0(x) \leqslant 1$ for all $x \in K_0$. Now put $\tilde{\phi} = (\alpha - \phi(x_0))\psi_0$. Then $\tilde{\phi}$ is the required extension.

(b) *implies* (a). Consider the continuous right action of S on $AP(S)$ defined by $(a,f) \to l_a f$. Let L be the subspace of $AP(S)$ consisting of constant functions, let K be the unit ball in $AP(S)$ and $\alpha = 1$. Let $a \in S$ be fixed, and define $\phi(f) = f(a)$ for each $f \in L$. Then ϕ is clearly invariant and $\phi(f) \leqslant 1$ for all $f \in L \cap K$. It is easy to see that any invariant extension $\tilde{\phi}$ of ϕ to $AP(S)$ such that $\tilde{\phi}(f) \leqslant 1$ for all $f \in K$ is a left-invariant mean on $AP(S)$.

A simple modification of the above argument also proves:

THEOREM 2. *The following conditions on a topological semigroup S are equivalent*:

(a) *$WAP(S)$ has a left-invariant mean.*

(b) *For any weakly almost periodic weakly continuous right action of S on a Hausdorff real topological vector space E, if L is an invariant subspace of E and K is a convex subset of E such that $K - x_0$ is invariant for some x_0 contained in $L \cap \operatorname{Int} K$, then for each invariant linear functional ϕ on L such that $\phi(x) \leqslant \alpha$ for all $x \in L \cap K$ and some fixed real number α, then there exists an invariant linear extension $\tilde{\phi}$ of ϕ to E such that $\tilde{\phi}(x) \leqslant \alpha$ for all $x \in K$.*

REMARKS. (1) Since $AP(S)$ and $WAP(S)$ always have a left-invariant mean for any group S (see [3, p. 38]) our Theorem 1 implies Theorem 1 in [1] and our Theorem 2 implies Theorem 3 in [1] without completeness hypothesis.

(2) If $aS \cap bS$ is nonempty for any $a, b \in S$, then S has property (b) of Theorem 1, since in this case $AP(S)$ has a left-invariant mean (see [4]).

(3) We have actually proved implicitly the following analogue of Theorem 15.A in Silverman [5]:

THEOREM 3. *The following conditions on S are equivalent*:

(a) *$AP(S)$ has a left-invariant mean.*

(b) *For any almost periodic continuous right action of S on a Hausdorff real topological vector space E, if p is a continuous sublinear functional on E such that $p(s \cdot x) \leqslant p(x)$ for each $s \in S$ and $x \in E$, and if ϕ is an invariant linear functional defined on a subspace F of E such that $\phi \leqslant p$, then there exists an invariant linear extension $\tilde{\phi}$ of ϕ to E such that $\tilde{\phi} \leqslant p$.*

A similar statement for $WAP(S)$ with "almost periodic continuous right action" replaced by "weakly almost periodic weakly continuous right action" also holds.

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