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PROPERTIES OF STANDARD MAPS

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ABSTRACT. Let X and Y be compact metric spaces. Let $S(X,Y)$ denote the collection of standard maps of X onto Y. We establish that $S(C,Y)$ is a dense subset of $C(C,Y)$, where C is the Cantor set. If f is a standard map and $G(f,Y)$ $\{A(f,Y)\}$ denotes the subgroup of $H(X)$ which preserves $\{$interchanges$\}$ the point-inverses of f, then there is a continuous homomorphism of $A(f,Y)$ into $H(Y)$ with kernel $G(f,Y)$. We also show that $G(f,Y)$ and $A(f,Y)$ are closed subsets of $H(X)$.

1. Let X and Y be topological spaces. Define $H(X,Y)$ $\{C(X,Y)\}$ to be the collection of homeomorphisms $\{$continuous maps$\}$ of X onto Y. If $X = Y$ we agree to write $H(X)$ $\{C(X)\}$. If $f \in C(X,Y)$, then $G(f,Y)$, defined by $\{h \in H(X)\colon f \circ h = f\}$, is a subgroup of $H(X)$. Let us say that f is a standard map if and only if (1) f is an identification and (2) $f(a) = f(b)$ implies there are sequences $x_n \in X$ and $h_n \in G(f,Y)$ satisfying $x_n \to a$ and $h_n(x_n) \to b$. Standard maps were first studied by A. Vobach [5]. In [5], Vobach showed that each compact metric space is the standard image of C, and the author [3], [4] has shown that each locally compact, separable (complete, separable) metric space is the standard image of $N \times C(P)$, where N denotes the positive integers and P the irrationals. The utility of these results is that the standard images of a given space are classified up to homeomorphism by the conjugacy classes of the group $G(f,Y)$ ([5], [3] and [4]).

2. In the sequel all spaces are compact and metrizable. We endow all function spaces with the compact-open topology.

Notation. $S(X,Y) = \{f \in C(X,Y)\colon f$ is a standard map$\}$. If $f \in C(X,Y)$ we shall denote the partition $\{f^{-1}(y)\colon y \in Y\}$ of X by $K(f)$.

DEFINITION. Let $f \in C(X,Y)$. We define the group $A(f,Y)$ to be $\{h \in H(X)\colon D \in K(f)$ implies $h(D) \in K(f)\}$.

REMARK. If $D \in K(f)$ and $h \in G(f,Y)$, then $h(D) = D$.

THEOREM 1. *$G(f,Y)$ is a normal subgroup of $A(f,Y)$ and there is a continuous homomorphism α which takes $A(f,Y)$ into $H(Y)$ with kernel $G(f,Y)$.*

PROOF. We establish the latter statement first. Let $h \in A(f,Y)$ and define

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$\alpha(h) = f \circ h \circ f^{-1}$. By a well-known theorem (see [1, p. 123]) $\alpha(h)$ is continuous if $\alpha(h)$ is single-valued. But this is clearly true since h preserves the fiber structure of f. Similarly, $[\alpha(h)]^{-1}$ is continuous and therefore $\alpha(h) \in H(Y)$.

α is continuous: Let (F,U) be any subbasic open subset of $H(Y)$. Then $\alpha^{-1}((F,U)) = (f^{-1}(F),f^{-1}(U)) \cap A(f,Y)$. Hence α is continuous.

α is a homomorphism: Let $h_1,h_2 \in A(f,Y)$. Then $\alpha(h_1 \circ h_2) = f \circ h_1 \circ h_2 \circ f^{-1} = (f \circ h_1 f^{-1}) \circ (f \circ h_2 \circ f^{-1})$ since h_1 and h_2 preserve the fiber structure of f. Thus, $\alpha(h_1 \circ h_2) = \alpha(h_1) \circ \alpha(h_2)$ as required.

kernel $\alpha = G(f,Y)$: Clearly, for each $h \in G(f,Y)$, $\alpha(h) = 1_Y$. If $\alpha(h) = 1_Y$, then, by definition, $f \circ h \circ f^{-1} = 1_Y$. For each $y \in Y$ we have $f(h(f^{-1}(y))) = y$ or, equivalently, $h(f^{-1}(y)) = f^{-1}(y)$. But $h \in A(f,Y)$ and hence $h(f^{-1}(y)) = f^{-1}(y)$. Therefore, $h \in G(f,Y)$ and $G(f,Y) =$ kernel α. This in turn implies the normality of $G(f,Y)$ in $A(f,Y)$.

Since Y is compact and metrizable the compact-open topology on $C(Y)$ coincides with the sup-metric topology. Moreover, the sup-metric d^+ on $C(Y)$ is a complete metric.

THEOREM 2. *$A(f,Y)$ and $G(f,Y)$ are closed subsets of $H(X)$.*

PROOF. To see that $A(f,Y)$ is closed we consider a limit point h of $A(f, Y)$ and a sequence $h_n \in A(f,Y)$ such that $h_n \to h$. Let α: $A(f,Y) \to H(Y) \subset C(Y)$ be the continuous homomorphism defined in Theorem 1. We will establish that $\alpha(h_n)$ converges to some element k of $H(Y)$ and that $\alpha(h) = k$.

Let p and p^+ $\{d$ and $d^+\}$ be the complete metrics of X and $C(X)$ $\{Y$ and $C(Y)\}$, respectively. Let $\varepsilon > 0$ be given. Since f is uniformly continuous there is a $\delta > 0$ such that $d(f(x),f(y)) < \varepsilon$ for each x and y satisfying $p(x,y) < \delta$. Now $h_n \to h$. Therefore, there is an integer M such that $n,m > M$ implies $p^+(h_n,h_m) < \delta$. Thus,

$$d(\alpha(h_n)(y),\alpha(h_m)(y)) = d\Big(f\big(h_n(f^{-1}(y))\big),f\big(h_m(f^{-1}(y))\big)\Big) < \varepsilon$$

provided $n,m > M$, y arbitrary. Hence $\alpha(h_n)$ is a Cauchy sequence in the complete space $C(Y)$. Let $\alpha(h_n) \to k \in C(Y)$. Then $f \circ h_n \circ f^{-1} \to k$ and $f \circ h_n \to k \circ f$. Therefore, $f \circ h = k \circ f$. In a similar way $f \circ h_n^{-1} \to f \circ h^{-1} = k' \circ f, k' \in C(Y)$. Hence $k \circ k' = 1_Y = k' \circ k$. Thus $k \in H(Y)$. $f \circ h \circ f^{-1} = k$ implies $h \in A(f,Y)$. We conclude $A(f,Y)$ is closed.

$G(f,Y)$ is the kernel of a continuous homomorphism and therefore is closed in $A(f,Y)$. But $A(f,Y)$ is closed in $H(X)$ and hence $G(f,Y)$ is closed in $H(X)$.

REMARK. The reader should note that Theorems 1 and 2 do not require the map f associated with $G(f,Y)$ and $A(f,Y)$ to be a standard map. In fact, Theorems 1 and 2 are true when f is any continuous map.

Let $f \in C(C,X)$, X a compact metric space. Clearly we always have the inclusion $G(f,X) \subset A(f,X)$. Since $A(f,X)$ describes a subgroup of $H(X)$, one might ask whether $A(f,X)$ describes anything other than $\{1_X\}$. In other words, is it true that $A(f,X) \neq G(f,X)$?

R. D. Anderson (see [2, p. 12]) has established the following theorem: *Let X be a compact metric space and $h \in H(X)$. Then there is a Cantor set C, $f \in C(C,X)$ and $\alpha \in H(X)$ such that $f \circ \alpha = h \circ f$.* This shows that f can be chosen from $C(C,X)$ in such a way that $A(f,X) \neq G(f,X)$ when $h \neq 1_X$. Using these techniques we prove in [3] that f can always be chosen from $S(C,X)$.

Our last theorem deals with the relationship of $S(C,X)$ to $C(C,X)$. Let us first establish the following two lemmas.

LEMMA 1. *Let X be homogeneous and $f \in S(Y,Z)$. Then $f \circ \pi: X \times Y \to Z$ is standard where π is the projection map on the Y-coordinate.*

PROOF. This follows immediately from the homogeneity of X and the standardness of f. The proof appears in [3].

LEMMA 2. *Let h be a homeomorphism of a Cantor set C_1 onto a Cantor set C_2. If $f \in S(C_2,X)$, then $f \circ h \in S(C_1,X)$.*

PROOF. The proof appears in [6].

THEOREM 3. *Let M be a compact metric space and C the standard "middle-thirds" Cantor set. If $f \in C(C,M)$ and $p \in S(C,M)$, then there is a $q \in S(C,C)$ such that $p \circ q \in S(C,M)$ and $d^+(f,p \circ q) < \varepsilon$, ε some preassigned positive number, and d^+ the sup-metric on $C(C,M)$.*

PROOF. Let C_j: $\{1 \leqslant j \leqslant n\}$ be a decomposition of C satisfying the following conditions:

(1) Each C_i is both open and closed,

(2) $C_i \cap C_j = \varnothing$, $i \neq j$, and

(3) diam $f(C_j) < \varepsilon$ for each i.

Define $E_j = p^{-1}(f(C_j))$ and $D_j = C \times E_j \times \{1/j\}$. Let $\mathfrak{D} = \cup\{D_j: 1 \leqslant j \leqslant n\}$ and note $D_j \simeq \mathfrak{D} \simeq C$. If $h \in G(p,M)$, then $h(E_j) = E_j$ for all j. Construct $\alpha \in H(C,\mathfrak{D})$ such that $\alpha(C_j) = D_j$ and define $q \in C(C,C)$ by $q = \pi \circ \alpha$ where π is the projection map on the E_j-coordinate, $\pi \in S(\mathfrak{D},C)$. If $\pi(c_1,e_1,1/j_1) = \pi(c_2,e_2,1/j_2)$, then $e_1 = e_2$ by definition of π. Let $x_n = (c_1,e_1,1/j_1)$ for each n and choose $h \in H(C)$, $k \in H(\{1/1,1/2,\ldots,1/n\})$ such that $h(c_1) = c_2, k(1/j_1) = 1/j_2$, respectively. Then $(h,1_C,k) \in G(\pi,C)$ and $(h,1_C,k)(x_n) \to (c_2,e_2,1/j_2)$ (equals in fact) as required. In view of Lemma 2, $\pi \circ \alpha \in S(C,C)$. Define $g \in C(C,M)$ by $g = p \circ q = (p \circ \pi) \circ \alpha$. By Lemma 2, $g \in S(C,M)$ if $p \circ \pi \in S(\mathfrak{D},M)$. But $p \circ \pi \in S(\mathfrak{D},M)$ by Lemma 1. Hence $g \in S(C,M)$.

Let $x \in C$ and $x \in C_j$ for some j. Consider $g(x) = p(\pi(\alpha(x)))$. Then $\alpha(x) \in D_j$ and $\pi(\alpha(x)) \in E_j$. Therefore, $p(\pi(\alpha(x))) \in f(C_j)$ which yields $d^+(f,p \circ q) < \varepsilon$.

COROLLARY 1. *$S(C,M)$ is a dense subset of $C(C,M)$.*

REMARK. Note that Theorem 3 actually proves that $\{p \circ S(C,C)\} \cap$

$S(C,M)$ is a dense subset of $C(C,M)$ for each $p \in S(C,M)$.

References

1. J. Dugundji, *Topology*, Allyn and Bacon, Boston, Mass., 1966. MR **33** #1824.

2. W. Gottschalk and J. Auslander (editors), *Topological dynamics*, Benjamin, New York, 1968. MR **38** #2762.

3. G. Huckabay, *On the classification of locally compact, separable metric spaces*, Fund. Math. (to appear).

4. ______, *Standard maps and the classification of topological spaces*, Ph.D. thesis, Oklahoma State University.

5. A. Vobach, *On subgroups of the homeomorphism group of the Cantor set*, Fund. Math. **60** (1967), 47–52. MR **36** #2108.

6. ______, *A theorem on homeomorphism groups and products of spaces*, Bull. Austral. Math. Soc. **1** (1969), 137–141. MR **39** #7586.

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