

***H*-SEMILOCAL DOMAINS AND ALTITUDE $R[c/b]$**

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ABSTRACT. It is shown that altitude $R[u] = \text{altitude } R$ holds for all u in the quotient field of a semilocal domain R such that $1/u$ is not in the Jacobson radical of the integral closure R' of R if and only if every height one prime ideal in R' has depth = altitude $R - 1$. Also, if (R, M) is a local domain, then every height one prime ideal p in $R[X]$ such that $p \subseteq (M, X)R[X]$ has depth = altitude R if and only if this holds for all such prime ideals which contain a linear polynomial.

1. Introduction. All rings in this paper are assumed to be commutative with an identity, and the undefined terminology is the same as that in [3].

This paper is concerned with two subjects which have previously been quite deeply investigated. The first of these is the altitude of the rings $R[c/b]$, where R is an integral domain and $b, c \in R$. (For two important papers on this subject, see [1], [10].) It turns out that, for given integers k and $a = \text{altitude } R$ such that $0 \leq k \leq 2a$, there exists a domain R and $b, c \in R$ such that altitude $R[c/b] = k$ [10, p. 606]. In this paper, we consider this problem for a semilocal (Noetherian) domain R . For such R , it is well known that if A is an arbitrary finite algebraic extension domain of R , then

$$\text{altitude } R \geq \text{altitude } A \geq \text{altitude } R - 1$$

(see (2.3)). Therefore, this holds, in particular, for the case $A = R[c/b]$. And in this case, if b/c is a nonzero element in the Jacobson radical of the integral closure R' of R , then altitude $A = \text{altitude } R - 1$. The main result in this paper, (2.8), shows that for all other elements in the quotient field of R ,

$$\text{altitude } R[c/b] = \text{altitude } R$$

if and only if the height one prime ideals in R' have a certain property (namely, that R' be an H -domain (2.5)). As an interesting and somewhat surprising corollary to this result, it is shown in (2.9) that if (R, M) is a local domain, then all height one prime ideals in $D = R[X]_{(M, X)}$ which contain a linear polynomial have depth = altitude R if and only if all height one prime ideals in D have this depth.

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The second subject with which this paper is concerned is H -domains. The concept of such rings has, in the past, been of some benefit when considering the chain conjectures. (For example, see [4], [5].) In §2 a number of characterizations (besides the main theorem and its just-mentioned corollary) of when the integral closure of a semilocal domain is an H -domain are given in (2.6), (2.10), and (2.11). One or more of these may, in the future, prove to be of help in deciding whether or not certain of the chain conjectures are true.

2. Main theorem. So as to avoid continual repetitions, the following notation is fixed throughout this section.

(2.1) *Notation.* R is a semilocal domain with quotient field F , altitude $R = a$, and R' is the integral closure of R in F . Also, let $\mathfrak{V} = \{V; V \text{ is a valuation ring in } F, R \subseteq V, \text{ and altitude } V = a\}$.

As was partially indicated in (2.1), we will frequently be interested in valuation over-rings of R throughout this section. The following remark gives two facts concerning such rings which will be of frequent use.

(2.2) **REMARK.** If $A \subseteq F$ is a finitely generated ring over R and altitude $A = d$, then there exists a valuation ring V in F such that $A \subseteq V$ and altitude $V = d$, by [3, (11.9)]. Moreover, if W is any valuation ring in F such that $A \subseteq W$, then altitude $W \leq d$ [2, Corollary 2, p. 67].

The following proposition is certainly known, but the author knows of no reference for it. Thus, since it is needed below, a sketch of its proof will be given.

(2.3) **PROPOSITION.** *If A is a finite algebraic extension domain of R , then altitude $R \geq$ altitude $A \geq$ altitude $R - 1$.*

PROOF. There exists a finite integral extension domain C of R such that $C \subseteq A$ and A and C have the same quotient field. Then altitude $R =$ altitude $C \geq$ altitude A , by (2.2) with C replacing R . Also, there exists $c \in C$ such that $A \subseteq C[1/c]$, so again by (2.2), altitude $C[1/c] \leq$ altitude A . Also, it readily follows from [4, Lemma 2.1] that altitude $C[1/c] \geq$ altitude $C - 1 =$ altitude $R - 1$. Q.E.D.

To prove the main theorem in this paper, we need the following three lemmas and a definition.

(2.4) **LEMMA.** *With the notation of (2.1), let $I_1 = \bigcap \{V; V \in \mathfrak{V}\}$, and let $I = \bigcap \{R'_p; p \in \text{Spec } R' \text{ and depth } p = a - 1\}$. Then $I_1 = I$.*

PROOF. If $a = 0$, then $I_1 = F = I$, and if $a = 1$, then $I_1 = R' = I$, so it may be assumed that $a > 1$.

Let $p \in \text{Spec } R'$ such that depth $p = a - 1$, and let $(0) \subset p = p_1 \subset \cdots \subset p_a$ be a maximal chain of prime ideals in R' of length $= a$ thru p . Then, by [3, (11.9)], there exists a valuation ring V in F such that $R' \subseteq V$ and V has a chain of prime ideals $(0) \subset P_1 \subset \cdots \subset P_a$ such that $P_i \cap R' = p_i$ ($i = 1, \dots, a$). Therefore altitude $V \geq a$, and altitude $V \leq a$ (2.2). Thus altitude $V = a$,

so $V \in \mathfrak{V}$ and $R'_p = V_{P_1} \supseteq V$ (since R' is a Krull domain, R'_p is a valuation ring), hence $I_1 \subseteq I$.

Now let $u = c/b$ ($b, c \in R'$) be an element in I . Then $u \in R'_p$, for all $p \in \text{Spec } R'$ such that $\text{depth } p = a - 1$, so $bR'_p : cR'_p = R'_p$, for all such prime ideals p in R' . Therefore, either $bR' : cR' = R'$ or $\text{depth } bR' : cR' < a - 1$. If $bR' : cR' = R'$, then $u \in R' \subseteq I_1$, as desired. Therefore assume that $u \notin R'$, so $\text{depth } bR' : cR' < a - 1$. Now, with $B = R'[u^{-1}]$, $B/u^{-1}B = R'/(bR' : cR')$ [6, Corollary 3 and Remark 8.2]. Thus, if N is a maximal ideal in B such that $u^{-1} \in N$, then $N = (M', u^{-1})B$, for some maximal ideal M' in R' and $\text{height } N/u^{-1}B < a - 1$ (since $\text{altitude } R'/(bR' : cR') = \text{depth } bR' : cR' < a - 1$). Therefore since B_N is integral over a local subdomain $L \supseteq A = R'[u^{-1}]$ (since B is integral over A and only finitely many maximal ideals in B lie over $N \cap A$ [3, (33.10)]),

$$\begin{aligned} a - 1 &> \text{height } N/u^{-1}B = \text{altitude } B_N/u^{-1}B_N \\ &= \text{altitude } L/(u^{-1}B_N \cap L) = \text{altitude } L/u^{-1}L \end{aligned}$$

(since $u^{-1}L$ and $u^{-1}B_N \cap L$ have the same minimal prime divisors [3, Example 4, p. 34]), and $\text{altitude } L/u^{-1}L = [3, (9.7)] \text{altitude } L - 1 = \text{altitude } B_N - 1$. Hence $\text{height } N < a$. Thus, if $V \in \mathfrak{V}$, then $B_N \not\subseteq V$ (2.2), so u^{-1} is not in the maximal ideal Q in V (since $Q \cap R'$ is a maximal ideal, by (2.2)). Therefore, since either u or u^{-1} is in V , it follows that $u \in V$, hence $u \in I_1$, so $I \subseteq I_1$. Q.E.D.

From now on, I will be used to denote the ring $I_1 = I$ in (2.4). (I has been quite deeply investigated in [8], and therein it is shown that it has many interesting properties.)

The following definition will be used in the next lemma and the remainder of this paper.

(2.5) DEFINITION. A ring A is an H -ring in case, for all height one prime ideals p in A , $\text{depth } p = \text{altitude } A - 1$.

H -rings were introduced in [4], and a number of properties of Noetherian H -rings were given in [4] and [5]. A brief summary of the known results which are related to the main theorem in this paper will be given following (2.8).

(2.6) LEMMA. *With the notation of (2.1), R' is an H -domain if and only if $R' = I$ (2.4).*

PROOF. Let \mathfrak{S} be the set of height one prime ideals in R' . Then, since R' is a Krull domain, $R' = \bigcap \{R'_p; p \in \mathfrak{S}\}$ and for every proper subset \mathfrak{S}' of \mathfrak{S} , $R' \subset \bigcap \{R'_p; p \in \mathfrak{S}'\}$. Therefore, R' is an H -domain if and only if every height one prime ideal in R' has $\text{depth} = a - 1$ if and only if $R' = I$ (by the definition of I in (2.4)). Q.E.D.

In regard to (2.6), it should be mentioned that it is an open problem whether there exists a local domain R such that R' is quasi-local and R' is not an H -domain. In fact, this is equivalent to the Chain Conjecture (the integral closure

of a local domain is catenary) [5, (2.4)]. Therefore, by (2.6), the Chain Conjecture is also equivalent to: if R' is quasi-local, then $R' = I$.

The following lemma will enable us to give a very short proof of (2.8).

(2.7) LEMMA. *With $J^* = \bigcap \{Q; Q \text{ is the maximal ideal in } V \in \mathfrak{V}\}$ and $J = \bigcap \{M'; M' \text{ is a maximal ideal in } R'\}$, $J^* = J$ if and only if R' is an H -domain. Moreover, $\{u \in F; \text{altitude } R[u] < a\} = \{1/x; 0 \neq x \in J^*\}$.*

PROOF. If $J^* = J$, then $J^* \subseteq R'$, so $I = R'$ (since R' is completely integrally closed and J^* is an ideal in I), hence R' is an H -domain (2.6).

For the converse, it will first be proved that $J^* \cap R' = \bigcap \{M' \in \text{Spec } R'; \text{height } M' = a\}$. For this, let $V \in \mathfrak{V}$, let Q be the maximal ideal in V , and let $M' = Q \cap R'$. Then $\text{height } M' = a$ (since $R'_{M'}$ is integral over a local domain $L \supseteq R$, hence

$$a = \text{altitude } V \leq (2.2) \text{altitude } L = \text{height } M' \leq a).$$

On the other hand, if M' is a maximal ideal in R' such that $\text{height } M' = a$, then, by [3, (11.9)], there exists a valuation ring (V, Q) in F such that $R' \subseteq V$, $Q \cap R' = M'$, and $\text{height } Q \geq a$. Therefore, since $R \subseteq V$,

$$\text{altitude } V \leq a \text{ (2.2),}$$

so $V \in \mathfrak{V}$. Thus

$$J^* \cap R' = \bigcap \{M' \in \text{Spec } R'; \text{height } M' = a\}.$$

Now if R' is an H -domain, then each maximal ideal M' in R' has $\text{height} = a$. (For R' has only finitely many maximal ideals, so if $b \in M'$ is such that b is not in any other maximal ideal in R' , and if p is a prime divisor of bR' , then $\text{depth } p = a - 1$ (by hypothesis) and M' is the only maximal ideal in R' which contains p , so $\text{height } M' \geq a = \text{altitude } R' \geq \text{height } M'$.) Therefore

$$J = \bigcap \{M' \in \text{Spec } R'; \text{height } M' = a\},$$

and $I = R'$ (2.6), so $J^* = J^* \cap R' = J$.

Finally, if $0 \neq x \in J^*$, then for each $V \in \mathfrak{V}$ (and with $u = 1/x$), $R'[u] \not\subseteq V$, so $\text{altitude } R[u] = \text{altitude } R'[u] < a$, by (2.2). For the opposite inclusion, if $u \in F$ and $\text{altitude } R[u] < a$, then there does not exist a $V \in \mathfrak{V}$ such that $u \in V$, by (2.2). Therefore $1/u$ is in the maximal ideal in V , for all $V \in \mathfrak{V}$, so $1/u \in J^*$. Q.E.D.

Since J^* is a proper ideal in I , the following known result is an immediate consequence of (2.7) (and (2.3)): for all $0 \neq u \in F$, either $\text{altitude } R[u] = a$ or $\text{altitude } R[1/u] = a$. But even more can be said when R' is not an H -domain. Namely, for all $x \in J^*$, $x \notin J$, $\text{altitude } R[1/x] = a - 1$ and $R[x]$ has at most h (and at least one) maximal ideals of $\text{height} = a$, where R has exactly h maximal ideals of $\text{height} = a$. This follows since $\text{altitude } R[x, 1/x] = a - 1$, so all maximal ideals of $\text{height} = a$ in $R[x]$ contain x and lie over a maximal

ideal M in R such that $\text{height } M = a$, by (2.2), so the maximal ideals in $R[x]$ of height $= a$ are of the form $(M, x)R[x]$, for some maximal ideal M in R such that $\text{height } M = a$.

We can now prove the main result in this paper. (I am indebted to the referee for his suggestion to use (2.7) to prove (2.8). My original proof of (2.8) was longer, and then (2.7) was given as a remark concerning the proof of the theorem.)

(2.8) THEOREM. *With the notation of (2.1), R' is an H -domain if and only if $\text{altitude } R[u] = \text{altitude } R$, for all $u \in F$ such that $1/u \notin J$, the Jacobson radical of R' .*

PROOF. By (2.7), R is an H -domain if and only if $J^* = J$ if and only if $\{u \in F; \text{altitude } R[u] < a\} = \{1/x; 0 \neq x \in J\}$. Q.E.D.

Since the hypothesis in (2.8) had to do with R' being an H -domain, it should be noted that if R' is an H -domain, then R is also an H -domain (by integral dependence). On the other hand, if R is an H -local domain, then either R' is an H -domain, or the height one prime ideals in R' which have depth $< a - 1$ are height one maximal ideals, by [5, (3.4)(1)] and [7, (4.5)]. (If R has more than one maximal ideal, then it is an open problem whether or not this last statement holds.) Now, if R is an H -local domain and R' is not an H -domain, then there exist $u \in F$ such that $\text{altitude } R[u] < \text{altitude } R$ and $1/u$ is not in the Jacobson radical of R' . In fact, each x which is such that x is in a maximal ideal M' in R' if and only if $\text{height } M' > 1$ is such that $\text{altitude } R[1/x] < \text{altitude } R$. (For an example of such R , let R be as in [3, Example 2, pp. 203–205] in the case $m = 0$.)

There are two results in the literature which are closely related to (2.8). The first says that a local domain (L, M) is an H -domain if and only if $\text{height } ML[c/b] = \text{altitude } L - 1$, for all analytically independent elements b, c in L [4, Proposition 4.7]. (From this it follows that if L is an H -domain, then $\text{altitude } L[c/b] = \text{altitude } L$, for all analytically independent elements $b, c \in L$.) The second related result says that a local domain (L, M) is such that its integral closure L' is an H -domain if and only if $L[X]_{(M, X)}$ is an H -domain [5, (3.2)]. (From this it follows that if L' is an H -domain, then $\text{altitude } L[c/b] = \text{altitude } L$, for all c/b such that $(M, c/b)L[c/b]$ is proper (which holds if and only if $b/c \notin L'$ [9, Lemma 2]). But, even if R is local, to prove the converse of (2.8) using [5, (3.2)] seems to be quite difficult.)

A number of corollaries of (2.8) will now be given. The first of these concerns the height one prime ideals in $R[X]$. Its proof uses [5, (3.2)], which was mentioned in the last paragraph, and its conclusion was a somewhat surprising result to the author.

(2.9) COROLLARY. *Let (R, M) be a local domain. Then R' is an H -domain if and only if every height one prime ideal $p \subseteq N = (M, X)R[X]$ which contains a linear polynomial is such that $\text{height } N/p = a$. If this holds and $q \subseteq Q$ are prime*

ideals in $R[X]$ such that $\text{height } q = 1$, Q is maximal, and $Q \cap R = M$, then $\text{height } Q/q = a$.

PROOF. Assume first that $\text{height } N/p = a$ whenever p is a height one prime ideal in $R[X]$ such that $p \subseteq N$ and p contains a linear polynomial. Let b, c be nonzero elements in R such that $b/c \notin J$, the Jacobson radical of R' . Then, to show that R' is an H -domain, it must be shown that $\text{altitude } R[c/b] = a$ (2.7). For this, note that $bX - c \in K = \text{Ker}(R[X] \rightarrow R[c/b])$ and $\text{height } K = 1$, so if $K \subseteq N$, then $\text{altitude } R[c/b] = a$, by hypothesis. If $K \not\subseteq N$, then $(M, c/b)R[c/b] = R[c/b]$, so $b/c \in R'$ [9, Lemma 2]. Now the condition implies that $\text{height}(M, u)R[u] = a$, for all nonunits $u \in R'$, so it follows that all maximal ideals in R' have $\text{height} = a$. Therefore, since $b/c \in R', \notin J, c/b \in R'_{M'}$, for some maximal ideal M' in R' , so

$$a \geq \text{altitude } R[c/b] = \text{altitude } R'[c/b] \geq \text{altitude } R'_{M'} = a.$$

Conversely, if R' is an H -domain, then $D = R[X]_N$ is an H -domain [5, (3.2)], so $\text{height } ND/pD = \text{altitude } D - 1 = (a + 1) - 1 = a$ for all height one prime ideals $p \subseteq N$. Moreover, if Q is a maximal ideal in $R[X]$ such that $Q \cap R = M$, then $R[X]_Q$ is also an H -domain [7, (5.1) and (5.3)], so the last statement is clear. Q.E.D.

Question. Does (2.9) generalize to irreducible polynomials of a given degree $n > 1$?

The following corollary extends (2.8) to finite algebraic extension fields. It also gives some indication that the above question may have an affirmative answer.

(2.10) COROLLARY. *With the notation of (2.1), let E be a finite algebraic extension field of F . Then R' is an H -domain if and only if $\text{altitude } R[e] = a$ for all $e \in E$ such that $1/e \notin J'$, the Jacobson radical of the integral closure R'' of R in E .*

PROOF. There exists a finite integral extension domain S of R such that E is the quotient field of S , and then S is a semilocal domain. Also, $\text{altitude } R[e] = \text{altitude } S[e]$, for all $e \in E$, and $R'' = S'$, the integral closure of S in E . Therefore, by (2.8), S' is an H -domain if and only if $\text{altitude } S[e] = a$ for all $e \in E$ such that $1/e \notin J'$. Finally, since height one prime ideals in S' lie over height one prime ideals in R' , by [3, (10.14)], it readily follows from integral dependence that S' is an H -domain if and only if R' is an H -domain. Q.E.D.

The final result in this paper gives two more characterizations of when R' is an H -domain.

(2.11) COROLLARY. *With the notation of (2.1), assume that (R, M) is local and let E be a finite algebraic extension field of F . Then the following statements are equivalent:*

(2.11.1) R' is an H -domain.

(2.11.2) For each $e \in E$ such that $MR[e] \neq R[e]$, every maximal ideal in $R[e]$ which lies over M has height $= a$.

(2.11.3) For each $e \in E$ such that $(M, e)R[e] \neq R[e]$, $\text{height}(M, e)R[e] = a$.

PROOF. Assume that (2.11.1) holds, let $e \in E$ such that $MR[e] \neq R[e]$, and let Q be a maximal ideal in $R[e]$ such that $Q \cap R = M$. Let

$$K = \text{Ker}(R[X] \rightarrow R[e])$$

and let N^* be the maximal ideal in $R[X]$ such that $K \subseteq N^*$ and $N^*/K = Q$. Then $R[X]_{N^*}$ is an H -domain, by [5, (3.2)] and [7, (5.1) and (5.3)], and $\text{height } K = 1$, so

$$\text{height } Q = \text{height } N^*/K = \text{height } N^* - 1 = a,$$

hence (2.11.2) holds.

It is clear that (2.11.2) \Rightarrow (2.11.3).

Finally, if (2.11.3) holds, then let p be a height one prime ideal in $R[X]$ which contains a linear polynomial. If $p \subseteq N = (M, X)R[X]$, then, with $e = X + p$, $\text{height } N/p = a$, by hypothesis, so R' is an H -domain (2.9). Q.E.D.

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