PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 64, Number 1, May 1977

RELATIONSHIPS BETWEEN CONTINUUM NEIGHBORHOODS IN INVERSE LIMIT SPACES AND SEPARATIONS IN INVERSE LIMIT SEQUENCES

HARVEY S. DAVIS

ABSTRACT. The main result of this paper is the following theorem. Let $\{X_\alpha, f_{\alpha\beta}, \alpha, \beta \in I\}$ be an inverse system of compact Hausdorff spaces and continuous onto maps with inverse limit X. Let $p \in X$ and A be closed in X. There exists a continuum neighborhood of p disjoint from A if and only if there exists $\alpha \in I$ and disjoint sets U and V open in X_α, neighborhoods respectively of p_α and A_α such that for all $\beta \geqslant \alpha, f_{\alpha\beta}^{-1}(U)$ lies in a single component of $X_\beta - f_{\alpha\beta}^{-1}(V)$. This is Theorem B of the text.

Introduction. Throughout this paper the conventions and elementary results on inverse limits of topological spaces are those given in Bourbaki [3] except when noted otherwise. $(I, \leqslant)$ is a directed set, $\{X_\alpha | \alpha \in I\}$ is a collection of nonvoid compact Hausdorff spaces and, for every pair (α, β) where $\alpha \leqslant \beta$, $f_{\alpha\beta}: X_\beta \to X_\alpha$ is continuous. The maps $f_{\alpha\beta}$ are not necessarily onto. $(X_\alpha, f_{\alpha\beta})$ is an inverse system of topological spaces with inverse limit X and canonical maps $f_\alpha: X \to X_\alpha$, for all α. If $A \subseteq X$, $p \in X$ then denote $f_\alpha(A) = A_\alpha$ and $f_\alpha(p) = p_\alpha$.

If A is any subset of a topological space S then $\mathfrak{T}(A)$ will denote the collection of all open (in S) subsets of A and $\mathfrak{F}(A)$ will denote the collection of all closed (in S) subsets of A. $\mathrm{Int}(A)$ will denote the interior of A and $\mathrm{Clo}(A)$ will denote the closure of A.

W is a *subcontinuum* of a topological space S if and only if W is compact, Hausdorff and connected. W is a *continuum neighborhood* of a point p of S if and only if p is an element of the interior of W. If A is any subset of S then $T(A)$ will denote the set of those points for which there does not exist a continuum neighborhood disjoint from A.

Let $\alpha \in I$. $\{W_\beta | \beta \geqslant \alpha\}$ denotes an *inverse system of subcontinua* if and only if for all $\beta \geqslant \alpha$, W_β is a subcontinuum of X_β, and $(W_\beta, f_{\alpha\beta}|W_\beta)$ is an inverse system.

Let S be a compact Hausdorff space. The following dictionary provides translations of some common properties of S into properties of the set function T defined on $\{A | A \subseteq S\}$.

Received by the editors September 29, 1975 and, in revised form, July 26, 1976.

AMS (MOS) subject classifications (1970). Primary 54F15; Secondary 54B25.

Key words and phrases. Continuum neighborhood, set function T, inverse limit, compact Hausdorff space.

S is *connected im kleinem* at $p \in S$ if and only if for every $A \in \mathfrak{F}(S)$, if $p \notin A$ then $p \notin T(A)$.

S is *semi-locally connected* at $p \in S$ if and only if $T(p) = p$.

S is *aposyndetic at q with respect to p* if and only if $q \notin T(p)$.

S is an *indecomposable continuum* if and only if for every $p \in S$, $T(p) = S$.

Other material on the set function T can be found in [1], [2], [4], [5].

A. **Fundamental theorems.** Theorems A1 and A2 establish necessary and sufficient conditions on the inverse limit sequence that a given point p is not an element of $T(A)$ for a given closed set A.

LEMMA A1. *Let $p \in X$ and $A \in \mathfrak{F}(X)$. If there exists $\alpha \in I$, $U \in \mathfrak{T}(X_\alpha)$, $V \in \mathfrak{T}(X_\alpha)$ and an inverse system of subcontinua $\{W_\beta | \beta \geqslant \alpha\}$ such that $p_\alpha \in U$, $A_\alpha \subseteq V$ and such that for all $\beta \geqslant \alpha$, $f_{\alpha\beta}^{-1}(U) \subseteq W_\beta \subseteq X_\beta - f_{\alpha\beta}^{-1}(V)$, then $p \notin T(A)$.*

PROOF. Let W be the canonical image of inv lim W_β in X. Since each W_β is a continuum, W is a continuum. Since, for each $\beta \geqslant \alpha$, $f_\alpha^{-1}(U) \subseteq \mathrm{Clo}(f_\alpha^{-1}(U)) \subseteq \mathrm{Clo}(f_\beta^{-1} f_{\alpha\beta}^{-1}(U)) \subseteq f_\beta^{-1}(W_\beta)$ and since $W = \bigcap \{f_\beta^{-1}(W_\beta) | \beta \geqslant \alpha\}$, $f_\alpha^{-1}(U) \subseteq W$. Since $p \in f_\alpha^{-1}(U)$, $p \in \mathrm{Int}(W)$. Since, for all $\beta \geqslant \alpha$,

$$f_\beta^{-1}(W_\beta) \subseteq X - f_\beta^{-1} f_{\alpha\beta}^{-1}(V) = X - f_\alpha^{-1}(V) = f_\alpha^{-1}(X_\alpha - V) \subseteq X - A,$$

$\bigcap \{f_\beta^{-1}(W_\beta) | \beta \geqslant \alpha\} \subseteq X - A$, so $W \cap A = \varnothing$. Therefore $p \notin T(A)$.

LEMMA A2. *Let $p \in X$ and $A \in \mathfrak{F}(X)$. If $p \notin T(A)$ then there exists $\alpha \in I$, $U \in \mathfrak{T}(X_\alpha)$, $V \in \mathfrak{T}(X_\alpha)$ and an inverse system of subcontinua $\{W_\beta | \beta \geqslant \alpha\}$ such that $p_\alpha \in U$, $A_\alpha \subseteq V$ and for all $\beta \geqslant \alpha$,*

$$f_{\alpha\beta}^{-1}(U) \cap f_\beta(X) \subseteq W_\beta \subseteq f_\beta(X) - f_{\alpha\beta}^{-1}(V).$$

PROOF. Since $p \notin T(A)$, there exists a subcontinuum W of X such that $p \in \mathrm{Int}(W)$ and $W \cap A = \varnothing$. There exists $\alpha(p) \in I$, $U(p) \in \mathfrak{T}(X_{\alpha(p)})$ such that $p \in f_{\alpha(p)}^{-1}(U(p)) \subseteq \mathrm{Int}(W)$ since $\{f_\alpha^{-1}(U) | \alpha \in I,\ U \in \mathfrak{T}(X_\alpha)\}$ is a basis for $\mathfrak{T}(X)$. Similarly, for each $x \in A$, there exists $\alpha(x) \in I$, $U(x) \in \mathfrak{T}(X_{\alpha(x)})$ such that $x \in f_{\alpha(x)}^{-1}(U(x)) \subseteq X - W$. Let $\{f_{\alpha(i)}^{-1}(U_i) | i = 1, \ldots, n\}$ be a finite subcover of the open cover $\{f_{\alpha(x)}^{-1}(U(x)) | x \in A\}$ of the compact set A. Since I is a directed set, there exists $\alpha \in I$ such that $\alpha \geqslant \alpha(p)$ and for $i = 1, \ldots, n$, $\alpha \geqslant \alpha(i)$. Let $U = f_{\alpha(p)\alpha}^{-1}(U(p))$ and $V = f_{\alpha(1)\alpha}^{-1}(U_1) \cup \cdots \cup f_{\alpha(n)\alpha}^{-1}(U_n)$. For $\beta \geqslant \alpha$ let $W_\beta = f_\beta(W)$.

Let $\beta \geqslant \alpha$ and let $z \in f_{\alpha\beta}^{-1}(U) \cap f_\beta(X)$. Since $z \in f_\beta(X)$, there exists $x \in X$ such that $x_\beta = z$. Since $f_{\alpha(p)\beta}(z) \in U(p)$, $x_{\alpha(p)} \in U(p)$. Therefore $x \in f_{\alpha(p)}^{-1}(U(p)) \subseteq \mathrm{Int}(W) \subseteq W$. Thus $z = x_\beta \in f_\beta(W) = W_\beta$. So $f_{\alpha\beta}^{-1}(U) \cap f_\beta(X) \subseteq W_\beta$. Now let $z \in W_\beta = f_\beta(W)$. Let $z = x_\beta$ for some $x \in W$. Clearly $z \in f_\beta(X)$. Suppose $z \in f_{\alpha\beta}^{-1}(V)$. There exists $i \leqslant n$ such that $f_{\alpha(i)\beta}(z) \in U_i$ so $x_{\alpha(i)} \in U_i$. Therefore $x \notin W$. This contradiction establishes that $W_\beta \subseteq f_\beta(X) - f_{\alpha\beta}^{-1}(V)$.

THEOREM A1. *Let $p \in X$ and $A \in \mathfrak{F}(X)$. The following are equivalent.*

(a) $p \notin T(A)$.

(b) *There exists $\alpha \in I$, $U \in \mathfrak{T}(X_\alpha)$, $V \in \mathfrak{T}(X_\alpha)$ and an inverse system of subcontinua $\{W_\beta | \beta \geqslant \alpha\}$ such that $p_\alpha \in U$, $A_\alpha \subseteq V$ and for all $\beta \geqslant \alpha$,*

$$f_{\alpha\beta}^{-1}(U) \cap f_\beta(X) \subseteq W_\beta \subseteq f_\beta(X) - f_{\alpha\beta}^{-1}(V).$$

PROOF. This follows directly from Lemmas A1 and A2 once it is noted that if W is the canonical image of inv lim W_β in X then for all β, $f_\beta(W) = f_\beta(X) \cap W_\beta$, $f_\beta(W)$ is a continuum and W is the canonical image of inv lim $f_\beta(W)$. The relations in Lemma A1 imply that $f_{\alpha\beta}^{-1}(U) \cap f_\beta(X) \subseteq f_\beta(W) \subseteq f_\beta(X) - f_{\alpha\beta}^{-1}(V)$.

THEOREM A2. *Suppose that each $f_{\alpha\beta}$ is onto. Let $p \in X$ and $A \in \mathfrak{F}(X)$. The following are equivalent.*

(a) $p \notin T(A)$.

(b) *There exists $\alpha \in I$, $U \in \mathfrak{T}(X_\alpha)$, $V \in \mathfrak{T}(X_\alpha)$ and an inverse system of subcontinua $\{W_\beta | \beta \geqslant \alpha\}$ such that $p_\alpha \in U$, $A_\alpha \subseteq V$ and such that for all $\beta \geqslant \alpha$,*

$$f_{\alpha\beta}^{-1}(U) \subseteq W_\beta \subseteq X_\beta - f_{\alpha\beta}^{-1}(V).$$

PROOF. This follows directly from Theorem A1 once it is noted that, since each X_α is compact, the hypothesis that each $f_{\alpha\beta}$ is onto forces each f_α to be onto.

COROLLARY A1. *Suppose that each $f_{\alpha\beta}$ is onto. Let J be a cofinal subset of I and let $A \in \mathfrak{F}(X)$. Then*

$$\cap \left\{ f_\alpha^{-1}(T(A_\alpha)) | \alpha \in J \right\} \subseteq T(A).$$

PROOF. Suppose $p \in X - T(A)$. Let α, U, V and $\{W_\beta\}$ be as in Theorem A2. Since J is cofinal, there exists $\beta \in J$ such that $\beta \geqslant \alpha$. But then $p_\beta \in f_{\alpha\beta}^{-1}(U) \subseteq W_\beta \subseteq X_\beta - f_{\alpha\beta}^{-1}(V) \subseteq X_\beta - A_\beta$. Therefore $p_\beta \notin T(A_\beta)$ so $p \notin f_\beta^{-1}(T(A_\beta))$. Thus $p \notin \cap \{f_\gamma^{-1}(T(A_\gamma)) | \gamma \in J\}$.

COROLLARY A2. *Suppose that each $f_{\alpha\beta}$ is onto. Let J be a cofinal subset of I and let $A \in \mathfrak{F}(X)$. If for all $\gamma, \delta \in I$ where $\gamma \leqslant \delta$, $f_{\gamma\delta}$ is monotone, then*

$$T(A) = \cap \left\{ f_\gamma^{-1}(T(A_\gamma)) | \gamma \in J \right\}.$$

PROOF. Suppose $p \in X - \cap \{f_\gamma^{-1}(T(A_\gamma)) | \gamma \in J\}$. There exists α such that $p_\alpha \notin T(A_\alpha)$ so there exist U, $V \in \mathfrak{T}(X_\alpha)$ and a continuum W such that $p_\alpha \in U \subseteq W \subseteq X_\alpha - V \subseteq X_\alpha - A_\alpha$. Now let $\beta \geqslant \alpha$ and let $W_\beta = f_{\alpha\beta}^{-1}(W)$. Since $f_{\alpha\beta}$ is monotone, W_β is a continuum. It is clear that $\{W_\beta | \beta \geqslant \alpha\}$ is an inverse system of subcontinua and also that $f_{\alpha\beta}^{-1}(U) \subseteq W_\beta \subseteq X_\beta - f_{\alpha\beta}^{-1}(V)$. Hence, by Theorem A2, $p \notin T(A)$ so $T(A) \subseteq \cap \{f_\gamma^{-1}(T(A_\gamma)) | \gamma \in J\}$. Com-

bining this with the above corollary gives the equality.

B. Separations in inverse limit sequences.

DEFINITION. Let M and N be subsets of a topological space S. *M separates N in S* if and only if there exist sets P and Q such that $S - M = P \cup Q$, $P \cap N \neq \emptyset \neq Q \cap N$ and $(P \cap \mathrm{Clo}(Q)) \cup (Q \cap \mathrm{Clo}(P)) = \emptyset$.

REMARKS. If $N \subseteq M$ then M does not separate N in S. If M does not separate N in S and $L \subseteq M$ then L does not separate N in S.

DEFINITIONS. Let $A \in \mathfrak{F}(X)$ and $p \in X$. $p \notin S(A)$ if and only if there exists $\alpha \in I$, $U \in \mathfrak{T}(X_\alpha)$ and $v \in \mathfrak{T}(X_\alpha)$ such that $p_\alpha \in U$, $A_\alpha \subseteq V$, $U \cap V = \emptyset$, and, for all $\beta \geqslant \alpha$, $f_{\alpha\beta}^{-1}(V)$ does not separate $f_{\alpha\beta}^{-1}(U)$ in X_β.

REMARK. Let $A \in \mathfrak{F}(X)$. It is immediate from the definitions that $A \subseteq S(A)$.

THEOREM B1. *Let $A \in \mathfrak{F}(X)$. $T(A) \subseteq S(A)$.*

PROOF. Suppose $p \notin S(A)$. Let α, U, V be as in the above definition of S. Now let $\beta \geqslant \alpha$. Since X_β is compact Hausdorff and $f_{\alpha\beta}^{-1}(V)$ does not separate $f_{\alpha\beta}^{-1}(U)$ in X_β, $f_{\alpha\beta}^{-1}(U)$ lies entirely in a single component W_β of $X_\beta - f_{\alpha\beta}^{-1}(V)$. It is clear that W_β is a subcontinuum of X_β, that $f_{\alpha\beta}^{-1}(U) \subseteq W_\beta$ and that $W_\beta \subseteq X_\beta - f_{\alpha\beta}^{-1}(V)$. For all $\beta \geqslant \alpha$, choose such a W_β. Now let $\gamma \geqslant \beta \geqslant \alpha$. Note that $f_{\beta\gamma}(W_\gamma) \subseteq f_{\beta\gamma}(X_\gamma - f_{\alpha\gamma}^{-1}(V)) \subseteq X_\beta - f_{\alpha\beta}^{-1}(V)$. Also note that $f_{\alpha\beta}^{-1}(U) = f_{\beta\gamma}(f_{\alpha\gamma}^{-1}(U)) \subseteq f_{\beta\gamma}(W_\gamma)$ so that $f_{\alpha\beta}^{-1}(U) \subseteq f_{\beta\gamma}(W_\gamma) \cap W_\beta$. Since W_γ is connected and $f_{\beta\gamma}(W_\gamma) \cap W_\beta \neq \emptyset$, $f_{\beta\gamma}(W_\gamma) \subseteq W_\beta$ so $\{W_\beta | f_{\beta\gamma}|_{W_\beta}\}$ is an inverse of subcontinua. Therefore by Lemma A1, $p \notin T(A)$. Thus $T(A) \subseteq S(A)$.

THEOREM B2. *Suppose that each $f_{\alpha\beta}$ is onto. Let $A \in \mathfrak{F}(X)$. $S(A) = T(A)$.*

PROOF. Suppose $p \notin T(A)$. Let α, U, V and $\{W_\beta\}$ be as in Theorem A2. Note that $A_\alpha \subseteq V$, $p_\alpha \in U$ and that $U \cap V = \emptyset$. Now let $\beta \geqslant \alpha$. Since $f_{\alpha\beta}^{-1}(U) \subseteq W_\beta$ and $W_\beta \cap f_{\alpha\beta}^{-1}(V) = \emptyset$, it is clear that $f_{\alpha\beta}^{-1}(V)$ does not separate $f_{\alpha\beta}^{-1}(U)$ in X_β. Hence $p \notin S(A)$ and thus $S(A) \subseteq T(A)$. By Theorem B1, $S(A) = T(A)$.

The main result can now be stated.

THEOREM B. *Suppose that each $f_{\alpha\beta}$ is onto. Let $A \in \mathfrak{F}(X)$, and $p \in X$. There exists a continuum neighborhood of p disjoint from A if and only if there exists $\alpha \in I$ and disjoint sets U and V open in X_α, neighborhoods respectively of p_α and A_α such that for all $\beta \geqslant \alpha$, $f_{\alpha\beta}^{-1}(U)$ lies in a single component of $X_\beta - f_{\alpha\beta}^{-1}(V)$.*

PROOF. Once it is noted that in a compact Hausdorff space the failure of an open set G to separate an open set D is equivalent to the existence of a component K of $S - G$ that contains D the theorem follows as a corollary to Theorem B2.

BIBLIOGRAPHY

1. D. P. Bellamy and H. S. Davis, *Continuum neighborhoods and filterbases*, Proc. Amer. Math. Soc. **27** (1971), 371–374. MR **43** #2653.

2. D. E. Bennett, *A characterization of locally connectedness by means of the set function T*, Fund. Math. **86** (1974), 137–141. MR **50** #11189.

3. N. Bourbaki, *General topology*. Parts 1 and 2, Addison-Wesley, Reading, Mass., 1966. MR **34** #5044a, 5044b.

4. H. S. Davis, D. P. Stadtlander and P. M. Swingle, *Properties of the set function T^n*, Portugal. Math. **21** (1962), 113–133. MR **25** #5501.

5. H. S. Davis, *A note on connectedness im kleinen*, Proc. Amer. Math. Soc. **19** (1968), 1237–1241. MR **40** #8021.

DEPARTMENT OF MATHEMATICS, MICHIGAN STATE UNIVERSITY, EAST LANSING, MICHIGAN 48824