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CONNECTED AND CONNECTIVITY MULTIFUNCTIONS

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ABSTRACT. Conditions are given under which multifunctions which preserve connected sets will have a connected graph.

Introduction. In [2] Long presented a condition under which a connected function is a connectivity map. The results in [2] required that both the domain and range spaces be first countable. In this note we generalize Long's condition so that the results are valid for multifunctions and we remove the requirement that the spaces be first countable.

A multifunction $F: X \to Y$ is *connected* if and only if for each connected set $M \subset X$, $F(M)$ is connected in Y. The graph map G for F is the multifunction $G: X \to X \times Y$ defined by $G(x) = \{x\} \times F(x)$. Then F is a connectivity multifunction if and only if G is connected. Further, the multifunction F is *upper semicontinuous* (u.s.c.) if and only if for each $x \in X$ and each open set $V \subset Y$ with $F(x) \subset V$ there is an open set U containing x such that $F(U) \subset V$. Also the multifunction F is called *point closed* (*compact*, *connected*) in case $F(x)$ is closed (compact, connected) for all $x \in X$.

If A is a subset of a topological space, then the closure of A is denoted by A^*.

The main results. Before proceeding to the principal result in this section, we present an example which shows that some theorems for connected functions cannot be extended to connected multifunctions. The following theorem appears in [2].

THEOREM A. *If $f: X \to Y$ is a connected function on a space X into the T_1-space Y and if M is any closed subset of Y, then each component of $f^{-1}(M)$ is closed.*

The following example shows that Theorem A is not true for connected multifunctions.

EXAMPLE. Let $X = [0, 1]$ and let Y be the unit square in the plane with corners (0, 0), (0, 1), (1, 1), (1, 0). Define $F(x) = \{(x, y): 0 \leqslant y \leqslant 1\}$ for $0 \leqslant x < 1$ and set $F(1) = \{(1, 1)\}$. If $M = [\frac{1}{2}, 1] \times [0, \frac{1}{2}]$, then $F^{-1}(M) = [\frac{1}{2}, 1)$, which is connected but not closed.

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There are a number of ways of extending the condition used by Long in [2]. Here we give the one that is useful in the result we wish to prove.

DEFINITION. Let $F: X \to Y$ and $y \in Y$. Then $x_0 \in T(F,y)$ if and only if for each open set $V \subset Y$ with $y \in V$ and each open set U with $x_0 \in U$, there exists an $x \in U$ such that $F(x) \cap V \neq \varnothing$.

In [2] Long obtained a characterization of continuity in terms of the set $T(f,y)$ for first countable spaces where f is single valued. We give an analogous characterization of upper semicontinuous multifunctions without assuming that X and Y are first countable. Note also that in Theorem 1 below the range space is assumed to be regular. In order to have unique convergence Long should have assumed that Y was T_2 in the first part of Theorem 3.4 of [2].

THEOREM 1. *Let $F: X \to Y$ be a point closed multifunction into the regular space Y. If F is u.s.c., then $x_0 \in T(F,y)$ implies that $y \in F(x_0)$. Moreover, if Y is compact and if $x_0 \in T(F,y)$ implies that $y \in F(x_0)$, then F is u.s.c.*

PROOF. Suppose that F is u.s.c. and that $y \notin F(x_0)$. Then there are disjoint open sets V_1, V_2 with $y \in V_1$ and $F(x_0) \subset V_2$. Further, since F is u.s.c., there is an open set U, with $x_0 \in U$, such that $F(U) \subset V_2$. Therefore if $x \in U$, $F(x) \cap V_1 = \varnothing$ and, hence, $x \notin T(F,y)$.

On the other hand suppose that Y is compact and that $x_0 \in T(F,y)$ implies that $y \in F(x_0)$. Now let V be an open set such that $F(x_0) \subset V$, and suppose that for each open set U containing x_0 there is an $x \in U$ such that $F(x) \not\subset V$. Then there is a net $\{x_\alpha ; \alpha \in D\}$ which converges to x_0 and such that there is a $y_\alpha \in F(x_0)\backslash V$ for each α. Then some subnet of $\{y_\alpha ; \alpha \in D\}$ converges to a point y_0 and, consequently, $x_0 \in T(F,y_0)$. But $y_0 \notin F(x_0)$, a contradiction. Hence, F is u.s.c.

Another property of the sets $T(F,y)$ is

THEOREM 2. *The sets $T(F,y)$ are closed.*

PROOF. If $x \notin T(F,y)$, then there are open sets U, V with $x \in U$ and $y \in V$ such that $F(U) \cap V = \varnothing$. But then $U \cap T(F,y) = \varnothing$. Thus $T(F,y)$ is closed.

We now give our main result.

THEOREM 3. *Let $F: X \to Y$ be a point compact, connected multifunction on the compact space X into the space Y. If for each connected set $M \subset X$ and for each $x \in M$, $T(F,y) \cap M^* = \{x\}$ for all $y \in F(x)$, then F is a connectivity multifunction.*

PROOF. Suppose there exists a connected set $M \subset X$ such that $G(M)$ is not connected. Then set $G(M) = H \cup K$ where H, K are nonempty separated sets in $X \times Y$. Now $\{x\} \times F(x)$ is compact and connected. Thus if $x \in G^{-1}(H)$, then $\{x\} \times F(x) \subset H$, and therefore $\{x\} \times F(x) \cap K^* = \varnothing$. Next let $A = G^{-1}(H)$ and $B = G^{-1}(K)$. We shall show that if $x \in A$, then $F(x)$ does not contain a limit point of $F(B)$. For this let $x_0 \in A$. Then by the above and the Wallace Theorem [1, Theorem 12, p. 142] there exist open sets U, V such that $x_0 \in U$, $F(x_0) \subset V$ and $U \times V \cap K^* = \varnothing$. Note if $x \in U \cap B$ and $y \in F(x)$, then $y \notin V$. Now suppose some $y_0 \in F(x_0)$ is a limit point of $F(B)$. Then there is a net y_α in $F(B) \cap V$ which converges to y_0. For each α let $b_\alpha \in B$ be such that $y_\alpha \in F(b_\alpha)$. From the above $b_\alpha \notin U$, and since $X \backslash U$ is compact, the net b_α has a subnet which converges to a point x_1 in $X \backslash U$. But then $x_1 \in T(F, y_0) \cap M^*$ which contradicts the hypothesis since $x_0 \in T(F, y_0) \cap M$ and $x_1 \neq x_0$. Therefore no element of $F(A)$ is a limit point of $F(B)$ and a dual argument shows that no point of $F(B)$ is a limit point of $F(A)$. Note that

$$\begin{aligned} F(M) = F(A \cup B) = F(A) \cup F(B) &= F(A) \cup [F(B) \backslash F(A)] \\ &= [F(A) \backslash F(B)] \cup F(B). \end{aligned}$$

Then if $F(A) \neq F(B)$ we have a separation of $F(M)$ which contradicts the hypothesis that F is connected. Finally, since M is connected, A and B are not separated. Suppose $x \in A \cap B^*$; then, as above, by the Wallace Theorem there are open sets U, V with $x \in U$, $F(x) \subset V$. Also $U \cap B \neq \varnothing$, and if $x' \in U \cap B$, $F(x) \neq F(x')$. Consequently, $F(M)$ is nondegenerate. Hence, if $F(A) = F(B) = F(M)$, $F(M) = (F(A) \backslash \{y\}) \cup \{y\}$ is a separation of $F(M)$ for any $y \in F(B)$. Thus $G(M)$ is connected and hence, F is a connectivity multifunction.

The following corollary follows immediately from Theorem 3 and generalizes Theorem 3.6 of [2].

COROLLARY. *Let $f: X \to Y$ be a connected function on a compact space X into a space Y. If for each connected subset $M \subset X$ and any $x \in M$, $T(f, f(x)) \cap M^* = \{x\}$, then f is a connectivity map.*

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