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ON THE INTERSECTION OF FREE FACTORS OF A FREE GROUP

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ABSTRACT. An example is given of a descending sequence of free factors of a free group F whose intersection is not a free factor of F.

It is known (see [1, Exercise 51(c), p. 49]) that the intersection of any set of direct summands of a countable free abelian group is again a direct summand. Verena Dyson has asked whether the corresponding statement is true for free factors of free groups. We give here a counterexample: there is in a free group F of countably infinite rank a descending sequence of free factors whose intersection is not a free factor. (It is known that the intersection of a finite number of free factors of F is again a free factor–this follows from the fact that if $K \leqslant A * B$ then $K \cap A$ is a free factor of K, which is in turn a consequence of the Kuroš subgroup theorem (see [2]). We mention also in passing that it is easy to find sequences of free factors of a free group whose union is not a free factor: Consider for example the subgroups of the free group on $x_1, x_2, \ldots,$ generated by the finite subsets of $\{x_1 x_2^2, x_2 x_3^2, x_3 x_4^2, \ldots\}$.)

We state our example as a theorem. We use the notation $[x,y]$ for the commutator $x^{-1}y^{-1}xy$.

THEOREM. *Let F be free on $\{a_1, a_2, \ldots; x_1, x_2, \ldots\}$. Write*

$$H = \langle x_1[a_1, x_2], x_2[a_2, x_3], \ldots, x_i[a_i, x_{i+1}], \ldots \rangle,$$

and for $i = 1, 2, \ldots,$

$$B_i = \langle x_1[a_1, x_2], \ldots, x_i[a_i, x_{i+1}], x_{i+1}, x_{i+2}, \ldots; a_{i+1}, a_{i+2}, \ldots \rangle.$$

Then the B_i are all free factors of F, $\bigcap_{i=1}^{\infty} B_i = H$, but H is not a free factor of F.

PROOF. It is easy to check that $B_i * \langle a_1, \ldots, a_i \rangle = F$, so that the B_i are all free factors of F.

We next show that H is not a free factor of F. Suppose the contrary; then the quotient of F by the normal closure of H, say G, is free. In any case it has the presentation

$$G = \langle a_1, a_2, \ldots; x_1, x_2, \ldots \mid x_1 = [x_2, a_1], \ldots, x_i = [x_{i+1}, a_i], \ldots \rangle.$$

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$$x_1 = [x_2, a_1] = [x_3, a_2, a_1] = \cdots = [x_i, a_{i-1}, \ldots, a_1] = \cdots,$$

it follows that x_1 is in the intersection of the lower central series of G. Since G is free this intersection is trivial, so that $x_1 = 1$ in G. We prove that this cannot be. To this end let g_1 be any nontrivial element of A_5, the alternating group on five letters. It is well known that every element of A_5 can be expressed as a commutator; thus $g_1 = [g_2, h_1]$ for some $g_2, h_1 \in A_5$. Define inductively $g_{i+1}, h_i \in A_5$ by choosing them arbitrarily subject to $g_i = [g_{i+1}, h_i]$. Then the map given by $x_i \to g_i$, $a_i \to h_i$, extends to a homomorphism from G to A_5, which sends x_1 onto $g_1 \neq 1$. Hence G cannot be free and H cannot be a free factor of F.

Finally we indicate why it is that $H = \bigcap B_i$. It is easy to verify that $\bigcap B_i \supseteq H$. For the reverse inclusion let w be any element of $\bigcap B_i$. There exists an n such that $w \in \langle a_1, \ldots, a_n; x_1, \ldots, x_n \rangle$. If one now tries to express w in terms of the free generators

$$x_1 a_1^{-1} x_2^{-1} a_1 x_2, \ldots, x_n a_n^{-1} x_{n+1}^{-1} a_n x_{n+1}, x_{n+1}, x_{n+2}, \ldots; a_{n+1}, a_{n+2}, \ldots,$$

of B_n, then it is not difficult to see that w can involve at most the first $n - 1$ of these generators, so that $w \in H$ as required.

References

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