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A CURVILINEAR EXTENSION OF IVERSEN-TSUJI'S THEOREM FOR SIMPLY CONNECTED DOMAIN

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ABSTRACT. Let D be a simply connected domain with at least two boundary points in the complex plane, and t a boundary point of D. For a meromorphic function $f(z)$ in D, $\limsup|f(z)|$ as $z \to t$ is given in terms of accessible boundary points and prime ends. This gives a curvilinear extension of Iversen-Tsuji's Theorem for a simply connected domain.

1. **Introduction**. Let D be an arbitrary domain in the complex domain, ∂D its boundary, E a compact subset of capacity zero on ∂D, and let t_0 be a point of E such that $N(t_0) \cap (\partial D - E) \neq \emptyset$ for every neighborhood $N(t_0)$ of t_0.

Iversen-Tsuji's Theorem [1, p. 16] states that if $f(z)$ is a meromorphic function in D and bounded in the intersection of D with some neighborhood of t_0, then

$$\limsup_{z\to t_0,\, z\in D} |f(z)| = \limsup_{b\to t_0,\, b\in\partial D} \Big(\limsup_{z\to b} |f(z)|\Big).$$

We give a curvilinear extension of this theorem for an arbitrary simply connected domain.

2. **Definitions**. A subset of the boundary of a simply connected domain D with at least two boundary points will be called a D-conformal null set if it corresponds to a set of linear measure zero under a one-to-one conformal mapping onto the unit disc. The set of all prime ends of D will be denoted by $\tilde{D}$. If $@$ is an accessible boundary point of D, then $@$ determines a unique prime end $P(@)$. The complex coordinate of an accessible boundary point $@$ will be denoted by $z(@)$.

Now we are ready to state our result.

3. THEOREM. *Let D be a simply connected domain in the complex plane, which is not the whole plane, t_0 a boundary point of D, $\tilde{E}$ a conformal null set of prime ends of D. If $f(z)$ is meromorphic in D and bounded in the intersection of D with some neighborhood of t_0, then*

$$(1)\qquad \limsup_{z\to t_0, z\in D} |f(z)| = \limsup_{z(@)\to t_0, P(@)\in \tilde{D}-\tilde{E}} \Big(\inf_{A}\Big(\limsup_{z\to z(@), z\in A} |f(z)|\Big)\Big),$$

where A is an arc at an accessible boundary point $@$ with $P(@) \in \tilde{D}-\tilde{E}$ and the convergence is in the sense of the ordinary euclidean metric.

PROOF. We may assume that $|f|$ is bounded by 1 (by multiplying a suitable

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constant). Let m = the right-hand side of (1). Then it suffices to show that $\limsup_{z\to t_0, z\in D}|f(z)| \leqslant m$.

Case 1. $m > 0$. Suppose the statement is not true. Then there exists a sequence $\{z_n\}$ in D converging to t_0 such that $|f(z_n)|$ converges to a positive constant c greater than m. Choose N so that $|f(z_n)| > m + (c - m)/2$ for all $n > N$.

Let ε be an arbitrary positive number less than $(c - m)/8$. Choose $r > 0$ so that at each accessible boundary point @ with $P(@) \in \tilde{D} - \tilde{E}$ and its complex coordinate in $\overline{D}_r \cap \partial D$ ($\overline{D}_r$ is the closure of $D_r = \{|z - t_0| < r\}$, and ∂D is the boundary of D), there exists an arc on which $|f(z)| < m + \varepsilon$.

Let G_n be the component of $D \cap D_r$ containing z_n $(n > N)$.

Let G_0 be the domain obtained by adding the ring domain $\{z: r/2 < |z - t_0| < 2r/3\}$ to the union of the G_n intersecting the ring domain (if $D \cap D_r$ has just one component, let G_0 be the component).

Let $\phi(t)$ be the real-valued function defined on ∂G_0 in the following way:

$$\begin{aligned}\phi(t) &= 0 \quad \text{for } t \in \partial G_0 \cap \{z: |z - t_0| < r/2\},\\ &= \log(2/m) \quad \text{elsewhere.}\end{aligned}$$

Let $u(z)$ be the harmonic function obtained from the Perron process with $\phi(t)$ and G_0. Let $g_n(w)$ be a one-to-one conformal mapping from the unit disc D_w onto G_n. Note that $f(g_n(w))$ is a bounded analytic function in D_w. Using Theorem 2.9 in [2, p. 30], $f(g_n(w))$ can be written as

$$f(g_n(w)) = w^m B(w) \exp(h(w)),$$

where $h(w)$ is an analytic function with negative real part in D_w, and $B(w)$ is the Blaschke product of zeros of $f(g_n(w))$. For each n we write

$$f(g_n(w)) = B_n(w) \exp(h_n(w)).$$

If $f(g_n(w))$ has no zero in D_w, let $B_n(w) \equiv 1$. Then the function $f(g_n(w))/B_n(w)$ is a bounded analytic function with no zero in D_w, and $\log|f(g_n(w))/B_n(w)|$ is a negative harmonic function. We note that $B_n(w)$ has a radial limit with modulus 1 almost everywhere on ∂D_w [2, Theorem 2.11, p. 32].

We consider the harmonic function

$$v(w) = \log|f(g_n(w))/B_n(w)| - u(g_n(w)), \qquad w \in D_w.$$

We know that for almost every $e^{i\theta}$ on ∂D_w, the image of the radius $te^{i\theta}$, $0 < t < 1$, determines an accessible boundary point of G_n. The function $v(w)$ has a radial limit at almost every point of ∂D_w.

If E_w is the set of all points $e^{i\theta}$ in ∂D_w satisfying the following conditions:

(1) the image of the radius $te^{i\theta}$, $0 < t < 1$, determines an accessible boundary point of G_n,

(2) $B_n(w)$ has a radial limit with modulus 1 at $e^{i\theta}$, and

(3) v has a radial limit at $e^{i\theta}$,

then $\partial D_w - E_w$ is of linear measure zero.

Let $e^{i\theta}$ be any point in E_{w^0}. If $e^{i\theta}$ corresponds to an accessible point @ with complex coordinate in $\partial G_n \cap \partial D \cap \{z: |z - t_0| < 3r/4\}$, and $P(@) \notin \tilde{E}$, $g_n^{-1}(A'_@)$ (where $A'_@$ is a suitable last part of $A_@$ contained in G_n) is an arc at $e^{i\theta}$ on which $|f(g_n(w))| < m + \varepsilon$. At almost every such point $e^{i\theta}$ the radial limit of $v(w)$ is not greater than $\log(m + \varepsilon)$. For, otherwise, we would have

$$\limsup_{w \to \partial D_w} v(w) \leqslant \log(m + \varepsilon) \quad \text{on } g_n^{-1}(A'_@)$$

and

$$\lim_{t \to 1} v(te^{i\theta}) > \log(m + \varepsilon);$$

thus $e^{i\theta}$ is an ambiguous point of the function $v(w)$. Hence there are only countably many such points by the Ambiguous Point Theorem (see [3] or [2, p. 85]).

If $e^{i\theta}$ corresponds to an accessible boundary point @ with complex coordinate $z(@)$ in $(\partial G_n - \partial G_n \cap \partial D) \cap \{z: |z - t_0| < 3r/4\}$, then since $\phi(t)$ is continuous at $z(@)$ and $z(@)$ is a regular point of ∂G, we have

$$\lim_{t \to 1} u\big(g_n(te^{i\theta})\big) = \phi(z(@)) = \log 2/m.$$

Hence we obtain

$$\limsup_{t \to 1} v(te^{i\theta}) \leqslant \log 1 - \log 2/m < \log m.$$

By Loewner's Lemma [1, p. 34], the set $\{e^{i\theta}: e^{i\theta} \in E_w, e^{i\theta}$ corresponds to an accessible point @ with $P(@) \in \tilde{E}\}$ is of linear measure zero. Hence altogether we have

$$\limsup_{t \to 1} v(te^{i\theta}) \leqslant \log(m + \varepsilon)$$

for almost every θ in $0 \leqslant \theta \leqslant 2\pi$.

Now let $v^*(w)$ be the harmonic conjugate of the harmonic function $v(w)$. Then the function

$$F(w) = \exp(v(w) + iv^*(w))$$

is a bounded analytic function in D_w. Therefore it has radial limits almost everywhere on ∂D_w.

Let C_w be the set of all points in ∂D_w in which $F(w)$ has a radial limit whose modulus is not greater than $m + \varepsilon$. Then $\partial D_w - C_w$ is of linear measure zero.

Since $F(w)$ can be represented by a Poisson integral in terms of the values $F(e^{i\theta})$ almost everywhere on ∂D_w, we have

$$F(w) = \frac{1}{2\pi} \int_{C_w} \frac{F(e^{i\theta})(1 - t^2)\, d\theta}{1 + t^2 - 2t\cos(\alpha - \theta)}.$$

Thus

$$|F(w)| \leqslant \frac{m + \varepsilon}{2\pi} \int_0^{2\pi} \frac{(1 - t^2)\, d\theta}{1 + t^2 - 2t\cos(\alpha - \theta)} = m + \varepsilon,$$

that is, $e^{v(w)} \leqslant m + \varepsilon$ in D_w. Therefore $v(w) \leqslant \log(m + \varepsilon)$ for every $w \in D_w$.

Let $w_n = g_n^{-1}(z_n)$. Then $v(w_n) \leqslant \log(m + \varepsilon)$, that is,

$$\log|f(g_n(w_n))/B_n(w)| - u(g_n(w_n)) \leqslant \log(m + \varepsilon).$$

Since t_0 is a regular boundary point of G_0, and $\phi(t)$ is continuous at t_0, we obtain

$$\lim_{n\to\infty} u(g_n(w_n)) = \lim_{n\to\infty} u(z_n) = 0.$$

Thus

$$\limsup_{n\to\infty} \log|f(g_n(w_n))/B_n(w)| \leqslant \log(m+\varepsilon).$$

Since $|B_n(w)| < 1$, we have

$$\limsup_{n\to\infty} \log|f(g_n(w_n))| \leqslant \log(m+\varepsilon).$$

Therefore

$$\limsup_{n\to\infty} |f(z_n)| \leqslant m+\varepsilon < m + (c-m)/8 < m + (c-m)/2,$$

contrary to our assumption.

Case 2. $m = 0$. Choose an arbitrary ε, $0 < \varepsilon < 1$, and replace m by ε. Then by the result in Case 1, we have

$$\limsup_{z\to t_0,\, z\in D} |f(z)| \leqslant \varepsilon.$$

Since ε was arbitrary, we have

$$\limsup_{z\to t_0,\, z\in D} |f(z)| \leqslant 0.$$

This completes the proof ot the Theorem.

As immediate consequences of the above Theorem we have the following corollaries.

COROLLARY 1. *Let D be a simply connected domain in the z-plane, which is not the whole plane, and t_0 a boundary point of D, $\tilde{E}$ a conformal null set of prime ends of D. If $f(z)$ is meromorphic in D and bounded in the intersection of D with some neighborhood $N(t_0)$ of t_0, and at each accessible boundary point @ with $P(@) \in \tilde{D} - \tilde{E}$, $z(@) \in \partial D \cap N(t_0)$, there exists an arc $A_@$ at $z(@)$ on which* $\limsup_{z\to z(@), z\in A_@} |f(z)| \leqslant m$, *then*

$$\limsup_{z\to t_0,\, z\in D} |f(z)| \leqslant m.$$

COROLLARY 2. *Let D be a simply connected domain in the z-plane, which is not the whole plane, and let t_0 be a boundary point of D, E a subset of ∂D such that the set $\{P(@)\colon z(@) \in E,$ @ is an accessible boundary point of $D\}$ is a D-conformal null set. If $u(z)$ is harmonic in D and bounded above in the intersection of D with some neighborhood of t_0, then*

$$\limsup_{z\to t_0, z\in D} u(z) = \limsup_{z(@)\to t_0, z(@)\in\partial D - E} \left(\inf_{A_@}\left(\limsup_{z\to z(@), z\in A_@} u(z)\right)\right),$$

where $A_@$ is an arc at an accessible boundary point @ with $z(@) \in \partial D - E$ and the convergence is in the sense of the ordinary euclidean metric.

COROLLARY 3. *If E is of $\frac{1}{2}$-dimensional Hausdorff measure zero, or of logarithmic capacity zero, on the boundary of D (in place of the assumption on E in Corollary 2), then the same conclusion holds as in Corollary 2.*

Note that E is a conformal null set [4].

COROLLARY 4 (BAGEMIHL [5]). *Let D be a simply connected domain which is not the whole plane and let $f(z)$ be an analytic function in D. Let $\tilde{E}$ be a set of prime ends which is mapped to a set of linear measure zero under a one-to-one conformal mapping of D onto a unit disc $D_w = \{w: |w| < 1\}$. Suppose that at every prime end P of D that does not belong to E there is a curve A_P such that* $\lim \sup_{z \to P, z \in A_P} |f(z)| \leqslant m$ *and that $f(z)$ does not have the asymptotic value ∞ at any prime end of D. Then it follows that* $\sup_{z \in D} |f(z)| \leqslant m$.

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