

A FRÉCHET ALGEBRA EXAMPLE

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ABSTRACT. A construction is given of a Fréchet algebra whose spectrum fails to be a k -space.

By the spectrum of a Fréchet algebra we mean the space of continuous, nonzero, complex-valued homomorphisms endowed with the Gelfand (i.e. relative weak*) topology. The kind of results discussed in [2], [3], [6] and [8] create interest in knowing whether such spectra are always k -spaces, i.e. whether the Gelfand topology is compactly generated. The purpose of this note is to show that, unfortunately, this is not always the case, and that, in fact, counterexamples may arise in a quite natural way.

Preliminaries. Let U be an open holomorphically convex subset of C^n . The algebra of holomorphic functions $\text{Hol}(U)$ is a Fréchet algebra with the compact-open topology, and the spectrum of this algebra is homeomorphic to U , where a point z in U is identified with the evaluation homomorphism $f \rightarrow f(z)$ (see [5, Corollary VII.A.7]). The fact that the Gelfand and Euclidean topologies for U coincide means that every neighborhood of a point w in U must contain a set of the form:

$$U(f_1, \dots, f_r; \varepsilon_1, \dots, \varepsilon_r) = \{z \in U: |f_j(z) - f_j(w)| < \varepsilon_j, j = 1, \dots, r\}$$

for some choice of f_1, \dots, f_r in $\text{Hol}(U)$ and positive numbers $\varepsilon_1, \dots, \varepsilon_r$. Of course, the f_j 's may be taken to be the coordinate functions, choosing $r = n$. The following shows that, in general, no fewer functions will suffice.

LEMMA. *Let U be an open subset of C^n , let $w \in U$, and suppose that f_1, \dots, f_r in $\text{Hol}(U)$ are such that for some positive numbers $\varepsilon_1, \dots, \varepsilon_r$, the set $U(f_1, \dots, f_r; \varepsilon_1, \dots, \varepsilon_r)$ has compact closure in U . Then $r \geq n$.*

PROOF. We shall use the following fact, which is contained in [5, Theorem 14, p. 115]:

(*) Let V_w be the germ of an analytic variety at w in C^n , and suppose that $\dim V_w = k$. Further, suppose that W is a neighborhood of w , and that $f \in \text{Hol}(W)$ vanishes at w . Let $Z = \{z \in W: f(z) = 0\}$. Then $\dim(V_w \cap Z_w) \geq k - 1$.

For $i = 1, \dots, r$, let $V_i = \{z \in U: f_j(z) - f_j(w) = 0, j = 1, \dots, i\}$, noting

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that each V_i is an analytic variety in U . Since $\dim C_w^n = n$ and $f_1 - f_1(w)$ vanishes at w , we may use (*) to obtain $\dim V_{1w} \geq n - 1$. Observing that $V_2 = V_1 \cap \{z \in U: f_2(z) - f_2(w) = 0\}$, we apply (*) again, obtaining $\dim V_{2w} \geq n - 2$. We continue in this way, and, after r applications of (*), we find that $\dim V_{rw} \geq n - r$. However, the variety V_r is contained in the relative closure of $U(f_1, \dots, f_r; \epsilon_1, \dots, \epsilon_r)$ which is compact. Hence, V_r is compact and must therefore consist of only finitely many points including w [5, p. 106]. There is then a neighborhood of w containing no other points of V_r , and this means that $\dim V_{rw} = 0 \geq n - r$, concluding the proof.

THE EXAMPLE. The construction given here can be carried out in considerably more generality, however, for the sake of simplicity we shall apply the Lemma only in the case $U = C^n$. (Also see [4].) Briefly, the strategy will be to take an inverse limit of algebras of entire functions of increasingly more variables.

More precisely, for each positive integer n , let $A_n = \text{Hol}(C^n)$, let q_n be the injection of C^n into C^{n+1} taking z to $(z, 0)$, and let $p_n: A_{n+1} \rightarrow A_n$ be the induced "restriction" mapping $f \rightarrow f \circ q_n$. The inverse limit system $\{A_n, p_n\}$ of Fréchet algebras and continuous homomorphisms gives rise to a limiting Fréchet algebra A and continuous dense homomorphisms $P_n: A \rightarrow A_n$ satisfying $P_n = p_n \circ P_{n+1}$ for all n (see [1, p. 170]). Let X be the spectrum of A , and for each n let $Q_n: C^n \rightarrow X$ be the continuous injection induced by P_n (i.e. the dual map $P_n': A'_n \rightarrow A'$ restricted to the spectra). Evidently Q_n maps each compact subset of C^n homeomorphically into X , and moreover: If K is a compact subset of X , then there is some m and compact subset K' of C^m such that $K = Q_m(K')$. This is an easy consequence of the fact that K is equicontinuous (see [7, Proposition 4.2]).

Now for each n , let U_n be the open unit ball in C^n , and define $W = \bigcup_{n=1}^\infty Q_n(U_n)$. Supposing that K is any compact subset of X , we find m and K' as above such that $K = Q_m(K')$. Then, since Q_m is injective and the sets $Q_n(U_n)$ are nested, we have $K \cap W = Q_m(K' \cap U_m)$. Moreover Q_m is a homeomorphism on K' , and $K' \cap U_m$ is open in K' , showing that $K \cap W$ is relatively open in K . However, W is not Gelfand-open. If it was a neighborhood of the point $x_0 = Q_1(0)$, then there would be elements a_1, \dots, a_r of A and positive numbers $\epsilon_1, \dots, \epsilon_r$, such that $\{x \in X: |\hat{a}_j(x) - \hat{a}_j(x_0)| < \epsilon_j, j = 1, \dots, r\} \subset W$. Suppose that $n > r$. Then the above inclusion forces the entire functions $P_n a_1, \dots, P_n a_r$ to satisfy

$$\{z \in C^n: |P_n a_j(z) - P_n a_j(0)| < \epsilon_j, j = 1, \dots, r\} \subset U_n,$$

contradicting the Lemma. Thus the Gelfand topology for X is not compactly generated.

Remarks. The topology generated by the Gelfand-compact subsets of X actually corresponds to the direct limit topology from the system $\{C^n, q_n\}$. More generally, whenever a Fréchet algebra is decomposed as an inverse limit of a countable system $\{A_n, p_n\}$ of Fréchet algebras (often Banach algebras in

practice), there is a corresponding direct limit system of spectra $\{X_n, q_n\}$. (See [4].) If each X_n is a k -space, then the spectrum X of A is a k -space iff the Gelfand and direct limit topologies coincide. An open question related to the Arens-Royden result for Fréchet algebras is whether these two topologies can have different first Čech cohomologies, see [2].

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