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SHORTER NOTES

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A CHARACTERIZATION OF METRIC COMPLETENESS

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ABSTRACT. A proof is given of a theorem, relevant to fixed-point theory, which implies that a metric space (X, d) is complete if and only if, for each continuous function h: $X \to \mathbf{R}$ bounded below on X, there is a point x_0 such that $h(x_0) - h(x) < d(x_0, x)$ for every other point x.

If (X, d) is a metric space and h is a function $X \to \mathbf{R}$, by a *d-point for h* we mean a point x_0 of X such that, for every other point x,

$$h(x_0) - h(x) < d(x_0, x).$$

In terms of this notion, the following theorem gives a necessary and sufficient condition for (X, d) to be complete. The necessity of the condition generalizes the proposition mentioned by Chi Song Wong in his recent note [1] concerning sufficient conditions for the existence of a fixed point for a function $X \to X$. (For some comparable results, and other methods of proof, see the papers listed in [1], especially those of Brønsted and Ekeland.)

THEOREM. *If the metric space (X, d) is complete then any lower semicontinuous function $X \to \mathbf{R}$ which is bounded below has a d-point. If (X, d) is not complete there is a uniformly continuous function $X \to \mathbf{R}$ which is bounded below but has no d-point.*

PROOF. Suppose first that (X, d) is complete, and let h be a function $X \to \mathbf{R}$ which is lower semicontinuous (with respect to d) and is bounded below. Taking x_1 to be any point of X, we choose a sequence $\{x_n\}$ in the following way. For each n, let

$$c_n = \inf\{h(x): h(x_n) - h(x) \geqslant d(x_n, x) > 0\},$$

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and let x_{n+1} be a point such that

$$h(x_n) - h(x_{n+1}) \geqslant d(x_n, x_{n+1}) \tag{1}$$

and

$$h(x_{n+1}) < c_n + n^{-1}. \tag{2}$$

(If x_n is a d-point for h then x_{n+1} must be x_n, and $c_n = \infty$.) From (1) it follows that the sequence $\{h(x_n)\}$ is nonincreasing, and that if $m \geqslant n$ then

$$h(x_n) - h(x_m) \geqslant d(x_n, x_m). \tag{3}$$

Since the sequence $\{h(x_n)\}$ is bounded below, it is convergent. Hence, by (3) and the assumed completeness, the sequence $\{x_n\}$ is convergent: let x_0 be its limit. Now

$$h(x_n) - h(x_0) \geqslant d(x_n, x_0) \tag{4}$$

for every n, because if, for some n,

$$h(x_n) - h(x_0) < d(x_n, x_0) - \varepsilon,$$

where $\varepsilon > 0$, then by the lower semicontinuity of h there would be a neighbourhood U of x_0 such that

$$h(x_n) - h(x) < d(x_n, x_0) - \varepsilon$$

for every x in U, and then m could be such that $x_m \in U$ and $d(x_m, x_0) < \varepsilon$, so that, contrary to (3),

$$h(x_n) - h(x_m) < d(x_n, x_0) - \varepsilon < d(x_n, x_m).$$

If x_0 is not a d-point for h then, for some x,

$$h(x_0) - h(x) \geqslant d(x_0, x) > 0. \tag{5}$$

From (4) (with $n + 1$ in place of n) and (2),

$$h(x) \leqslant h(x_{n+1}) + h(x) - h(x_0) < c_n + n^{-1} + h(x) - h(x_0).$$

Hence, by (5), we can choose n so that $h(x) < c_n$. From (4) and (5), $h(x_n) > h(x)$, so that $x_n \neq x$ and therefore $d(x_n, x) > 0$, and, moreover,

$$h(x_n) - h(x) \geqslant d(x_n, x).$$

It now follows from the definition of c_n that $h(x) \geqslant c_n$, and we have a contradiction. Thus x_0 is a d-point for h.

Now suppose that (X, d) is not complete, and let $\{x_n\}$ be a Cauchy sequence (with respect to d) which is not convergent. For any point x of X, $\{2d(x, x_n)\}$

is a Cauchy sequence in $\mathbf{R}$: let $h(x)$ be its limit. Then $h(x) > 0$, so the function h is bounded below. Also, if $x_0 \in X$,

$$|h(x_0) - h(x)| \leqslant 2d(x_0, x),$$

so h is uniformly continuous; and

$$\tfrac{1}{2}\{h(x_0) + h(x)\} \geqslant d(x_0, x),$$

so

$$h(x_0) - h(x) \geqslant d(x_0, x) + \tfrac{1}{2}\{h(x_0) - 3h(x)\}.$$

Now, by the definition of h, $h(x_m) \to 0$ as $m \to \infty$; therefore $3h(x) < h(x_0)$ if $x = x_m$ and m is large. Thus x_0 is not a d-point for h.

REMARKS. (i) When $X = \mathbf{R}$, and d is the usual metric for $\mathbf{R}$, a function $X \to \mathbf{R}$ which is uniformly continuous but not bounded below may have, but need not have, a d-point.

(ii) When d and h are given, a relation $\ll$ can be defined on X by the stipulation that $x \ll y$ if and only if $h(y) - h(x) \geqslant d(x,y) > 0$. This relation orders X (being transitive, antisymmetric, and strictly irreflexive), and it is clear that a point of X is a d-point for h if and only if it is minimal with respect to $\ll$.

(iii) If f is a function $X \to X$, it may be possible to choose d and h so that the relation $\ll$ has the poperty that if $f(x) \neq x$ then $f(x) \ll x$ (or, more generally, $y \ll x$ for some y), and then any d-point for h is a fixed point for f. Thus the first part of the above theorem yields the class of fixed-point theorems considered in [1]. In special cases, h can be appropriately defined in terms of d and f by the formula $h(x) = \beta d(x, f(x))$, with suitable values for the constant β. For example, if d and f satisfy Banach's condition

$$d(f(x), f(y)) \leqslant \alpha d(x,y) \quad \text{for all } x, y,$$

where $\alpha < 1$, we can take $\beta = (1 - \alpha)^{-1}$; and if (as in [2])

$$d(f(x), f(y)) \leqslant \alpha\{d(x, f(y)) + d(y, f(x))\} \quad \text{for all } x, y,$$

where $\alpha < \frac{1}{2}$, we can take $\beta = (1 - 2\alpha)^{-1}(1 - \alpha)$. (In each of these cases, it can be assumed that X is the closure of the f-orbit of some point.)

REFERENCES

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