

## NONMETRIZABLE HEREDITARILY LINDELÖF SPACES WITH POINT-COUNTABLE BASES FROM CH

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**ABSTRACT.** A nonmetrizable hereditarily Lindelöf space with a point-countable base is obtained by using the continuum hypothesis to construct a Lusin subspace of a countable chain condition first countable nonseparable Baire space.

There has lately been much interest in the relationship between hereditary Lindelöfness and hereditary separability within the class of regular spaces. See e.g. [Ru<sub>2</sub>]. We shall assume all spaces regular. The question is whether either property entails the other, possibly with other conditions on the spaces, possibly with the assumption of various consistent set-theoretic axioms. A rule of thumb which has emerged from recent investigations is the kinds of counterexamples constructed from a Souslin line can also be obtained from the assumption of the continuum hypothesis (henceforth called CH). For example, a Souslin line is hereditarily Lindelöf nonseparable, while such spaces were later obtained using CH [HJ]. Similarly, hereditarily separable non-Lindelöf spaces were first constructed from a Souslin line [Ru<sub>1</sub>], then from CH [HJ]. We provide another confirmation of this rule by using CH to construct a hereditarily Lindelöf space with a point-countable base which is not metrizable (and hence not separable). Such spaces were obtained from Souslin lines in [P] and [B]. See [T<sub>1</sub>] for further background.

Our construction may be outlined as follows. A *Lusin space* is an uncountable space such that every nowhere dense set is countable. We observe that dense Lusin subspaces of (first countable) countable chain condition nonseparable spaces are (first countable) hereditarily Lindelöf nonseparable. As in [T<sub>1</sub>], a first countable example may be converted into one with a point-countable base. By examining the classical proof (see e.g. [Ku]) that CH entails the existence of Lusin subspaces of the real line, we see that such subspaces exist in countable chain condition (henceforth CCC) first countable spaces without isolated points satisfying the Baire category theorem. The proof is completed

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by exhibiting a nonseparable example of the latter type of space.

First we consider Lusin spaces.

**DEFINITION.** A space is *Baire* if no nonempty open set is the union of countably many nowhere dense sets.

**DEFINITION.**  $\pi(X)$  is the least cardinal of a collection (called a  $\pi$ -base) of nonempty open sets such that every nonempty open set includes one.  $c(X)$  is the sup of cardinals of collections of disjoint open subsets of  $X$ .

**THEOREM 1.** *Assume CH. If  $X$  is uncountable CCC Baire without isolated points and  $\pi(X) \leq 2^{\aleph_0}$ ,  $X$  includes a dense Lusin subspace.*

**COROLLARY.** *Assume CH. Every uncountable CCC first countable Baire space without isolated points includes a dense Lusin subspace.*

**PROOF.** The Corollary follows since CCC first countable spaces have cardinality and hence weight  $\leq 2^{\aleph_0}$  [J]. To prove the theorem, first observe that every nowhere dense subset of  $X$  is included in the complement of the union of a maximal disjoint collection of elements of the  $\pi$ -base. Each such complement is nowhere dense and there are  $\leq (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0}$  of them, say  $\{F_\alpha\}_{\alpha < \omega_1}$ . If we can construct a dense uncountable  $Y \subseteq X$  with countable intersection with each  $F_\alpha$ , we are done. Let  $\{P_\alpha\}_{\alpha < \omega_1}$  be a  $\pi$ -base for  $X$ . Suppose for each  $\beta < \alpha < \omega_1$ , a  $y_\beta$  has been chosen in  $P_\beta - \bigcup_{\gamma < \beta} F_\gamma$ . Pick

$$y_\alpha \in P_\alpha - \left( \bigcup_{\beta < \alpha} F_\beta \cup \{y_\beta : \beta < \alpha\} \right).$$

Then  $\{y_\alpha\}_{\alpha < \omega_1}$  is dense Lusin.

Note that this argument does not require the  $P_\alpha$ 's to be distinct and thus encompasses the case when  $X$  is the real line. If  $X$  is CCC nonseparable, the isolated points clause can be dropped if the Lusin set is not required to be dense. As the theorem stands, the isolated points clause is necessary:

**EXAMPLE.** A locally compact first countable Hausdorff space with a countable dense set of isolated points and no Lusin subspace.

Let  $X$  consist of the real line plus those points in the upper half plane with both coordinates rational. The rational points are isolated. For each point on the line choose a fixed sequence of rational points converging to it and let neighbourhoods of the point be tails of the sequence.

(We have recently learned that Theorem 1 and other interesting results concerning Lusin spaces were announced in [AŠ]. White [Wh] was the first to note Lusin subspaces yielded hereditarily Lindelöf spaces.)

It is perhaps of interest that a variation of the conclusion of Theorem 1 is actually equivalent to CH.

**THEOREM 2.** *CH iff every uncountable CCC Baire space of  $\pi$ -weight  $\leq 2^{\aleph_0}$  without isolated points has a Lusin subspace of power  $2^{\aleph_0}$ .*

**PROOF.** Assuming the right-hand side, both the real line with the usual topology and the real line with the *density topology* [T<sub>3</sub>] have Lusin subspaces

of power continuum. In the latter topology, a Lusin subspace is a *Sierpiński set*, i.e. a set of reals such that every subset of measure zero is countable. By [R], the simultaneous existence of a Lusin subspace of the reals and a Sierpiński set of reals, at least one of power continuum, implies CH.

Now to construct the required Baire space.

**THEOREM 3.** *There exists a first countable zero-dimensional CCC Baire space in which no nonempty open subset is separable.*

**CONSTRUCTION.** The example is the space  $\mathcal{K}$  of all nonempty compact nowhere dense subsets of  $\mathbf{R}$  (the real line), equipped with the Pixley-Roy topology (defined below). This topology for a space of subsets is studied in [vD<sub>1</sub>] and is motivated by an example of Pixley and Roy [PR]. Because points, or subsets, of  $\mathcal{K}$  are subsets, or families of subsets of  $\mathbf{R}$ , they are denoted by Roman capitals, or script capitals, respectively; families of subsets of  $\mathcal{K}$  are denoted by German capitals.

For any subsets  $F$  and  $G$  of  $\mathbf{R}$  we define a subset  $[F, G]$  of  $\mathcal{K}$  by

$$[F, G] = \{K \in \mathcal{K} : F \subseteq K \subseteq G\}.$$

The Pixley-Roy topology on  $\mathcal{K}$  is defined by taking for each  $K \in \mathcal{K}$ , all sets of the form  $[K, U]$ , where  $U$  is a neighbourhood of  $K$  in  $\mathbf{R}$ , to be a neighbourhood base for  $K$  in  $\mathcal{K}$ . Since each  $K \in \mathcal{K}$  is a compact subset of  $\mathbf{R}$ , it is easily verified that  $\mathcal{K}$  is first countable. It is also easy to verify that each set of the form  $[F, G]$  is closed in  $\mathcal{K}$ . Since  $\{K\} = [K, K]$  for  $K \in \mathcal{K}$ , it follows that  $\mathcal{K}$  is a zero-dimensional  $T_1$ -space.

Recall that a family of sets is *centered* if each finite subfamily has nonempty intersection. We call a family  $\sigma$ -centered if it is the union of countably many centered subfamilies. In order to show that  $\mathcal{K}$  is a CCC space, we prove the stronger result that  $\mathcal{K}$  has a  $\sigma$ -centered base. In [vD<sub>2</sub>] it is shown that this is equivalent to  $b\mathcal{K}$  being separable for some (equivalently: for every) Hausdorff compactification  $b\mathcal{K}$  of  $\mathcal{K}$ . Let  $\mathcal{W}$  be any countable base for  $\mathbf{R}$  such that  $V \cup W \in \mathcal{W}$  for any  $V, W \in \mathcal{W}$ . Then for each compact  $K \subseteq \mathbf{R}$  and each neighbourhood  $U$  of  $K$  in  $\mathbf{R}$  there is a  $W \in \mathcal{W}$  such that  $K \subseteq W \subseteq U$ . So, if we define for each  $W \in \mathcal{W}$  a collection  $\mathcal{C}(W)$  by

$$\mathcal{C}(W) = \{[K, W] : K \in \mathcal{K} \text{ and } K \subseteq W\},$$

then the family  $\mathcal{C} = \cup \{\mathcal{C}(W) : W \in \mathcal{W}\}$  is a base for  $\mathcal{K}$ . But  $\mathcal{C}$  is  $\sigma$ -centered since  $U \cap V \in \mathcal{C}(W)$  whenever  $U, V \in \mathcal{C}(W)$ , and  $\emptyset \notin \mathcal{C}(W)$ , for each  $W \in \mathcal{W}$ .

In order to prove that  $\mathcal{K}$  is a Baire space, it suffices to show that  $\mathcal{K}$  satisfies the following "completeness condition", which resembles Čech completeness:

There is a sequence  $\{\mathcal{B}_n : n < \omega\}$  of families of (closed) subsets of  $\mathcal{K}$  satisfying:

(1) for each  $n < \omega$ , for each  $K \in \mathcal{K}$  and for each neighbourhood  $\mathcal{U}$  of  $K$  in  $\mathcal{K}$ , there is a  $\mathcal{B} \in \mathcal{B}_n$  such that  $K \subseteq \text{int}_{\mathcal{K}} \mathcal{B}$  and  $\mathcal{B} \subseteq \mathcal{U}$ ; and

(2) if  $\mathfrak{A} \subseteq \cup \{\mathfrak{B}_n: n < \omega\}$  is any centered family such that  $\mathfrak{A} \cap \mathfrak{B}_n \neq \emptyset$  for each  $n < \omega$ , then  $\cap \mathfrak{A} \neq \emptyset$ .

If  $\{B_n: n < \omega\}$  is any base for  $\mathbf{R}$ , then we define the families  $\mathfrak{B}_n$  by

$$\mathfrak{B}_n = \{[K, F]: K \in \mathfrak{K}, F \subseteq \mathbf{R} \text{ is compact, } K \subseteq \text{Int}_{\mathbf{R}} F, B_n - F \neq \emptyset\}.$$

Then (1) is an immediate consequence of the fact that each member of  $\mathfrak{K}$  is a compact nowhere dense subset of  $\mathbf{R}$ . Next let  $\mathfrak{A} = \{[K_i, F_i]: i \in I\}$  be any centered subfamily of  $\cup \{\mathfrak{B}_n: n < \omega\}$  such that  $\mathfrak{A} \cap \mathfrak{B}_n \neq \emptyset$  for each  $n < \omega$ . Define  $F = \cap \{F_i: i \in I\}$ . Then for each  $n < \omega$  we have that  $B_n - F \neq \emptyset$  since  $\mathfrak{A} \cap \mathfrak{B}_n \neq \emptyset$ . Therefore  $F$  is a nowhere dense subset of  $\mathbf{R}$  which clearly is compact. Next we observe that for any  $i, j \in I$  we have that  $K_i \subseteq F_j$ , since

$$\emptyset \neq [K_i, F_i] \cap [K_j, F_j] = [K_i \cup K_j, F_i \cap F_j].$$

Therefore  $K_i \subseteq F$  for all  $i \in I$ . But this shows that  $F \neq \emptyset$ , hence  $F \in \mathfrak{K}$ , and also that  $K_i \subseteq F \subseteq F_i$  for all  $i \in I$ , so  $F \in \cap \mathfrak{A}$ .

We now show that no nonempty open subset of  $\mathfrak{K}$  is separable. Let  $K$  be any member of  $\mathfrak{K}$ , let  $U$  be any open subset of  $\mathbf{R}$  which contains  $K$ , and let  $\mathfrak{D}$  be a dense subset of  $[K, U]$ . For each  $x \in U$  the set  $K \cup \{x\}$  belongs to  $\mathfrak{K}$ , hence  $K \cup \{x\} \subseteq D \subseteq U$  for some  $D \in \mathfrak{D}$ . Therefore  $U = \cup \mathfrak{D}$ . Since each member of  $\mathfrak{D}$  is nowhere dense, it follows that  $\mathfrak{D}$  is not countable.

It is of interest to note that  $\mathfrak{K}$  is submetrizable (i.e.  $\mathfrak{K}$  has a weaker metrizable topology). Indeed, let  $\mathfrak{S}$  be the family consisting of all sets of the form  $\{K \in \mathfrak{K}: K \cap U \neq \emptyset\}$  or  $\{K \in \mathfrak{K}: K \subseteq U\}$ , where  $U$  is open in  $\mathbf{R}$ . Each member of  $\mathfrak{S}$  is open in  $\mathfrak{K}$ , but  $\mathfrak{S}$  is a subbase for the Vietoris topology on  $\mathfrak{K}$ . This topology is second countable and hence metrizable [M]. It follows that  $\mathfrak{K}$  has a  $G_\delta$ -diagonal and a countable separating open cover. Hodel [H] has shown that a regular quasi-complete CCC Baire space  $X$  is separable provided it has a  $G_\delta$ -diagonal or a point-countable separating open cover and notes that the requirement that  $X$  be Baire is necessary.  $\mathfrak{K}$  establishes that quasi-completeness is also essential.

$\mathfrak{K}$  is in fact *cometrizable* [AP]. This observation is used in [T<sub>4</sub>] to construct, assuming Martin's Axiom plus not CH, a dense Baire normal noncollection-wise Hausdorff subspace of  $\mathfrak{K}$  having caliber  $\aleph_1$ .

We now have a CCC nonseparable first countable 0-dimensional Baire space  $\mathfrak{K}$  without isolated points. Let  $\mathfrak{L}$  be a dense Lusin subspace of  $\mathfrak{K}$ . Then  $\mathfrak{L}$  is CCC nonseparable. It is easily verified that CCC spaces in which nowhere dense sets are Lindelöf are hereditarily Lindelöf. As in [T<sub>1</sub>], take  $Z = \{z_\alpha\}_{\alpha < \omega_1}$ ,  $z_\alpha \notin \{z_\beta: \beta < \alpha\}$ . Take a neighbourhood base  $\{U_{\alpha, i}\}_{i < \omega}$  for  $z_\alpha$  such that for each  $\alpha$  and each  $i$ ,  $U_{\alpha, i} \cap \{z_\beta: \beta < \alpha\} = \emptyset$ . Then  $\{U_{\alpha, i}: \alpha < \omega_1, i < \omega\}$  is a point-countable base for  $Z$ , which thus is the desired nonmetrizable 0-dimensional hereditarily Lindelöf space with a point-countable base.

$Z$  also has a  $G_\delta$ -diagonal. The example in [T<sub>1</sub>] does not have a  $G_\delta$ -diagonal

because it is linearly orderable, and a linearly orderable space with a  $G_\delta$ -diagonal is metrizable [L].

It should be remarked that it was the requirement that our space be first countable that occasioned some effort to find an example. We can give a quick proof from CH of the existence of a hereditarily Lindelöf 0-dimensional nonseparable space by constructing a Lusin subspace of  $\{f \in 2^{\omega_1} : f(\alpha) = 0 \text{ except for countably many } \alpha\}$ . The latter space (with the subspace topology inherited from the product) is easily seen to be CCC, nonseparable, 0-dimensional, Baire, and to have  $\pi$ -weight  $\aleph_1$ . See e.g. [T<sub>2</sub>]. Other spaces that are CCC, nonseparable, Baire, and have  $\pi$ -weight  $\leq 2^{\aleph_0}$  are, for example, the density topology and the Stone space of the reduced measure algebra.

We take this opportunity to mention a couple of results that, if observed earlier, would have appeared in [T<sub>1</sub>].

**THEOREM 3.** *If there is a countably compact perfectly normal space which is not hereditarily separable, then there is a nonmetrizable hereditarily Lindelöf space with a point-countable base.*

**PROOF.** Countably compact perfectly normal spaces are first countable, and by [St] have no uncountable discrete subspace. Apply Theorem 2 of [T<sub>1</sub>].

**THEOREM 4.** *If there is a nonmetrizable hereditarily Lindelöf space with a point-countable base, there is a 0-dimensional one. If there is a nonseparable hereditarily Lindelöf space, there is a 0-dimensional one.*

**PROOF.** If CH, our example will do. If not CH, any completely regular space of cardinality  $\aleph_1$  is 0-dimensional. It is easy to see that if either of the spaces mentioned in the hypotheses of the theorem exists, then it has a subspace of power  $\aleph_1$  with the same properties.

Jensen [DJ] has constructed a model of set theory in which there are no Souslin lines, but CH holds. In this model there is a nonmetrizable hereditarily Lindelöf space with a point-countable base, but there is none which is linearly ordered.

Under the assumption of Martin's Axiom plus not CH, every countably compact perfectly normal space is hereditarily separable [W]. It remains open whether this assumption ensures the metrizability of hereditarily Lindelöf spaces with point-countable bases. However, after completing this paper, we learned that Kunen [K] had recently proved

**THEOREM 5.** *If Martin's Axiom plus not CH is assumed, there are no CCC Lusin spaces.*

In the light of another observation of Kunen—namely that Souslin lines have dense Lusin subspaces—the construction of a nonmetrizable hereditarily Lindelöf space with a point-countable base from a Souslin line can be regarded as a special case of the method developed in this paper.

Subsequent to our work, Kunen constructed from CH a compact perfectly

normal nonseparable space. By Theorem 3 this yields another nonmetrizable hereditarily Lindelöf space with a point-countable base.

Using the new axiom BACH, the second author [T<sub>5</sub>] has constructed a dense Lusin subspace of  $2^\kappa$ , for regular  $\kappa < 2^{\aleph_1}$ .

Subspaces of the resulting hereditarily Lindelöf nonseparable space all have weight either  $\kappa$  or  $\leq \aleph_1$ .

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