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THE NORM OF THE SUM OF TWO PROJECTIONS

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ABSTRACT. Let $\mathbf{e}$ and $\mathbf{f}$, $\mathbf{e} + \mathbf{f} \neq 0$, be two projections of a C^*-algebra $\mathbf{A}$. J. Duncan and P. J. Taylor have shown that $\|\mathbf{e} + \mathbf{f}\| = 1 + \|\mathbf{ef}\|$. In this paper an algebraic proof of this equality is given.

Let $\mathbf{A}$ be a C^*-algebra with unit 1, and let $\mathbf{e}$ and $\mathbf{f}$ be two projections of $\mathbf{A}$, not both zero. J. Duncan and P. J. Taylor have shown [2, Theorem 7] that

$$\|\mathbf{e} + \mathbf{f}\| = 1 + \|\mathbf{ef}\|. \tag{1}$$

Their proof is geometric. It is based upon a formula of Chandler Davis [1], [3] giving a matrix representation of two projections in generic position in a Hilbert space. Here we shall give an algebraic proof of the above equality.

THEOREM. *The spectrum* $\sigma(\mathbf{e} + \mathbf{f})$ *of* $\mathbf{e} + \mathbf{f}$ *is contained, with possible exception of the point* $\lambda = 0$, *in the interval* $[1 - \|\mathbf{ef}\|, 1 + \|\mathbf{ef}\|]$. *The point* $\lambda = 1 + \|\mathbf{ef}\|$ *belongs to* $\sigma(\mathbf{e} + \mathbf{f})$. *Hence* (1) *holds.*

PROOF. Denote by $\mathbf{r(ef)}$ the spectral radius of $\mathbf{ef}$. From the inequalities $(n > 1)$

$$\|\mathbf{efe}\|^n = \|(\mathbf{efe})^n\| = \|(\mathbf{ef})^n e\| \leqslant \|(\mathbf{ef})^n\| = \|(\mathbf{efe})^{n-1}\mathbf{f}\| \leqslant \|\mathbf{efe}\|^{n-1}$$

we deduce that $\mathbf{r(ef)} = \|\mathbf{efe}\| = \|\mathbf{ef}\|^2$.

Now, let λ be any complex number $\neq 0, 1$. Since

$$\begin{aligned}(\lambda - 1 + \mathbf{e})(\lambda - \mathbf{e} - \mathbf{f})(\lambda - 1 + \mathbf{f}) &= \left[(\lambda - 1)(\lambda - \mathbf{f}) - \mathbf{ef}\right](\lambda - 1 + \mathbf{f}) \\ &= \lambda\left[(\lambda - 1)^2 - \mathbf{ef}\right],\end{aligned}$$

we have

$$\left[(\lambda - 1)^2 - \mathbf{ef}\right]^{-1} = \lambda(\lambda - 1 + \mathbf{f})^{-1}(\lambda - \mathbf{e} - \mathbf{f})^{-1}(\lambda - 1 + \mathbf{e})^{-1}.$$

Hence

$$\lambda(\lambda - \mathbf{e} - \mathbf{f})^{-1} = (\lambda - 1 + \mathbf{f})\left[(\lambda - 1)^2 - \mathbf{ef}\right]^{-1}(\lambda - 1 + \mathbf{e}). \tag{2}$$

Because $(\lambda - \mathbf{e})(\lambda - 1 + \mathbf{e}) = \lambda(\lambda - 1)$ we get from (2)

$$\lambda(\lambda - 1)^2\left[(\lambda - 1)^2 - \mathbf{ef}\right]^{-1} = (\lambda - \mathbf{f})(\lambda - \mathbf{e} - \mathbf{f})^{-1}(\lambda - \mathbf{e}). \tag{3}$$

It follows that $(\lambda - \mathbf{e} - \mathbf{f})^{-1}$ exists for $\lambda \neq 0, 1$ iff $[(\lambda - 1)^2 - \mathbf{ef}]^{-1}$ exists.

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This implies that the spectrum $\sigma(\mathbf{ef})$ of $\mathbf{ef}$ is real and belongs to the interval $[0, \mathbf{r}(\mathbf{ef})] = [0, \|\mathbf{ef}\|^2]$. Let $\lambda \in \sigma(\mathbf{e} + \mathbf{f})$, $\lambda \neq 0$. It follows from (2) that $(\lambda - 1)^2 \in \sigma(\mathbf{ef}) \in [0, \|\mathbf{ef}\|^2]$. Therefore, $1 - \|\mathbf{ef}\| \leqslant \lambda \leqslant 1 + \|\mathbf{ef}\|$. Since $\lambda = \mathbf{r}(\mathbf{ef}) = \|\mathbf{ef}\|^2 \in \sigma(\mathbf{ef})$, we have $1 + \|\mathbf{ef}\| \in \sigma(\mathbf{e} + \mathbf{f})$. Hence, $\|\mathbf{e} + \mathbf{f}\| = 1 + \|\mathbf{ef}\|$. Q.E.D.

REMARK 1. Let $\lambda \in \sigma(\mathbf{e} + \mathbf{f})$, $\lambda \neq 0, 1, 2$. Since $(\lambda - 1)^2 \in \sigma(\mathbf{ef})$, it follows from (3) that $2 - \lambda \in \sigma(\mathbf{e} + \mathbf{f})$. Therefore, the set $\sigma(\mathbf{e} + \mathbf{f}) \cap (1 - \|\mathbf{ef}\|, 1 + \|\mathbf{ef}\|)$ is symmetric with respect to the point $\lambda = 1$.

REMARK 2. Put $(\lambda - \mathbf{ef})^{-1} = \mathbf{u}$ and $(\lambda - \mathbf{efe})^{-1} = \mathbf{v}$. If $\mathbf{v}$ exists then $\mathbf{ev} = \mathbf{ve}$. From $(\lambda - \mathbf{ef})(1 + \mathbf{evf}) = \lambda - \mathbf{ef} + \mathbf{e}(\lambda - \mathbf{efe})\mathbf{vf} = \lambda$ and

$$(1 + \mathbf{evf})(\lambda - \mathbf{ef}) = \lambda - \mathbf{ef} + \mathbf{ev}(\lambda - \mathbf{efe})\mathbf{f} = \lambda$$

we have $\lambda\mathbf{u} = 1 + \mathbf{evf}$. Similarly we get $\lambda\mathbf{v} = 1 + \mathbf{efue}$. This implies that $\sigma(\mathbf{ef}) = \sigma(\mathbf{efe})$. Further, it follows from (2) and (3) that $(\lambda - \mathbf{e} - \mathbf{f})^{-1}$ can be expressed by $[(\lambda - 1)^2 - \mathbf{efe}]^{-1}$, and vice versa.

REMARK 3. If $\|\mathbf{ef}\| < 1$, then $\lambda = 0$ is an isolated point of $\sigma(\mathbf{e} + \mathbf{f})$ (or possibly $\lambda = 0$ belongs to the resolvent set of $\mathbf{e} + \mathbf{f}$). In this case the right-hand side of (2) is analytic at $\lambda = 0$ with value $(1 - \mathbf{f})(1 - \mathbf{ef})^{-1}(1 - \mathbf{e}) = \mathbf{g}$ there. Hence, $\lambda^{-1}\mathbf{g}$ is the principal part of the resolvent $(\lambda - \mathbf{e} - \mathbf{f})^{-1}$ at $\lambda = 0$. The coefficient $\mathbf{g}$ is the projection of $\mathbf{e} + \mathbf{f}$ corresponding to the isolated point $\lambda = 0$ of $\sigma(\mathbf{e} + \mathbf{f})$. If we put $\lambda = 2$ in (3) we get $2(1 - \mathbf{ef})^{-1} = (2 - \mathbf{f})(2 - \mathbf{e} - \mathbf{f})^{-1}(2 - \mathbf{e})$. Hence we have also

$$\mathbf{g} = 2(1 - \mathbf{f})(2 - \mathbf{e} - \mathbf{f})^{-1}(1 - \mathbf{e}).$$

Suppose now that $\mathbf{e}$ and $\mathbf{f}$ are projections acting in a Hilbert space $\mathbf{H}$. Let $\mathbf{H}_1 = \mathbf{eH}$ and $\mathbf{H}_2 = \mathbf{fH}$ be the corresponding base spaces. Then $\mathbf{g}\xi = 0$, $\xi \in \mathbf{H}$, holds if and only if $\mathbf{e}\xi + \mathbf{f}\xi = 0$, and $\mathbf{e}\xi + \mathbf{f}\xi = 0$ holds if and only if $\mathbf{e}\xi = \mathbf{f}\xi = 0$, i.e., $\xi \in \mathbf{H}_1^{\perp} \cap \mathbf{H}_2^{\perp}$. Hence $\mathbf{g}$ is the projection onto $\mathbf{H}_1^{\perp} \cap \mathbf{H}_2^{\perp}$. It follows that $2\mathbf{f}(\mathbf{e} + \mathbf{f})^{-1}\mathbf{e}$ is the projection onto $\mathbf{H}_1 \cap \mathbf{H}_2$, if $(\mathbf{e} + \mathbf{f})^{-1}$ exists.

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