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A NOTE ON TWO CONGRUENCES ON A GROUPOID

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ABSTRACT. Let S be a groupoid and θ_p, θ_m the congruences on S defined as follows: $x\theta_p y$ ($x\theta_m y$) iff every prime (minimal prime) ideal of S containing x contains y and vice versa. It is proved that θ_p is the smallest congruence on S for which the quotient is a semilattice. It is also shown that S/θ_m is a disjunction semilattice if S has 0 and is a Boolean algebra if S is intraregular and closed for pseudocomplements. Some connections between the ideals of S and those of the quotients are established. Congruences similar to θ_p and θ_m are defined on a lattice using lattice-ideals; quotients under these are distributive lattices.

Introduction. This paper consists of four sections. §1 is devoted to a summary of some results on ideals and pseudocomplements which are used in subsequent sections. In §2 we define two congruences on a groupoid S in terms of prime ideals. The quotients under these are semilattices. Relations between the ideals of S and those of the quotients are investigated. Theorems 3 and 6 are the main theorems in this direction. §3 deals with the additional properties of the congruences. Two congruences on a lattice are given in §4. Some connected results for commutative semigroups and lattices have been proved by Kist [3], Varlet [5] and Venkatanarasimhan [6]. For standard concepts and notations used in this paper, the reader may refer to Birkhoff [1] and Clifford and Preston [2].

1. **Preliminaries**. By an ideal we mean an ideal in the sense of [2] and by a lattice-ideal we mean an ideal in the sense of [1]. Let S be a groupoid. A nonempty subset A of S is called a filter if $a, b \in A \Leftrightarrow ab \in A$. The smallest filter containing an element a is called the principal filter generated by a and is denoted by $K(a)$. The principal ideal generated by a is denoted by $J(a)$. An ideal $|A|$ of S is said to be prime if $ab \in A \Rightarrow a \in A$ or $b \in A$. S is said to be intraregular if $J(ab) = J(a) \cap J(b)$ for any two elements a, b of S. If S has 0 and $a \in S$, by the pseudocomplement of a we mean an element a^* of S such that $aa^* = 0 = a^*a$ and $ab = 0 = ba \Rightarrow ba^* = b = a^*b$. We shall denote lattice-join and lattice-meet by the symbols $\vee$ and $\wedge$ respectively.

We need the following lemmas in the sequel.

LEMMA I [4]. *A nonempty proper subset A of a groupoid S is a prime* (*minimal*

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prime) ideal of S if and only if $S - A$ is a filter (maximal filter) of S.

LEMMA II [4]. *Any prime ideal (proper filter) of a groupoid with 0 contains (is contained in) a minimal prime ideal (maximal filter).*

LEMMA III [4]. *A groupoid S is intraregular if and only if every ideal of S is an intersection of prime ideals.*

LEMMA IV [4]. *In an intraregular groupoid S with 0, a prime ideal is minimal prime if and only if it contains precisely one $J(x)$, $J(x)^*$ for every $x \in S$. In an intraregular groupoid S closed for pseudocomplements, a prime ideal is minimal prime if and only if it contains precisely one of x, x^* for every $x \in S$.*

LEMMA V [4]. *In an intraregular groupoid with 0, the pseudocomplement of an ideal is the intersection of all the minimal prime ideals not containing it.*

LEMMA VI [7]. *A semilattice S with 0 is a disjunction semilattice if and only if distinct principal ideals of S have distinct pseudocomplements.*

LEMMA VII [7]. *A disjunction semilattice closed for pseudocomplements is a Boolean algebra.*

LEMMA VIII (cf. [2, p. 12, Example 1]). *Let f be a homomorphism of a groupoid S onto a groupoid T. Then:*

(i) *If A is an ideal of S, $f(A)$ is an ideal of T. If B is an ideal of T, $f^{-1}(B)$ is an ideal of S.*

(ii) *If A is a prime ideal (filter) of T, $f^{-1}(A)$ is a prime ideal (filter) of S.*

(iii) *$f(J(x)) = J(f(x))$ for all $x \in S$.*

Proofs are not given for the results on filters since they may be inferred from the ones for prime ideals using Lemma I.

2. Two congruences on a groupoid.

Throughout this section, S will denote a groupoid.

For $x, y \in S$ define $x\theta_p y$ ($x\theta_m y$) to mean that every prime (minimal prime) ideal of S containing x contains y and vice versa. It is easily seen that θ_p and θ_m are congruences on S and that S/θ_p and S/θ_m are semilattices. We shall denote the natural homomorphism of S onto S/θ_p (S/θ_m) by θ_p^* (θ_m^*).

LEMMA 1. *Let A be a prime ideal or a filter (minimal prime ideal or a maximal filter) of S. Then $\theta_p^{*-1}\theta_p^*(A) = A$ ($\theta_m^{*-1}\theta_m^*(A) = A$).*

PROOF. Clearly the prime (minimal prime) ideals of S are unions of θ_p (θ_m)-classes. Hence the lemma follows.

THEOREM 1. *If A is any prime ideal (filter) of S, then $\theta_p^*(A)$ is a prime ideal (filter) of S/θ_p. If A is a minimal prime ideal (maximal filter) of S, then $\theta_p^*(A)$ is a minimal prime ideal (maximal filter) of S/θ_p and $\theta_m^*(A)$ is a minimal prime ideal (maximal filter) of S/θ_m.*

PROOF. Let A be a prime ideal of S. Clearly, $x, y \in S$, $\theta_p^*(x)\theta_p^*(y) \in \theta_p^*(A)$

$\Rightarrow xy \in A \Rightarrow x \in A$ or $y \in A \Rightarrow \theta_p^*(x) \in \theta_p^*(A)$ or $\theta_p^*(y) \in \theta_p^*(A)$. Hence $\theta_p^*(A)$ is a prime ideal.

If A is a minimal prime ideal and B is any prime ideal of S/θ_p such that $B \subseteq \theta_p^*(A)$, then $\theta_p^{*-1}(B) \subseteq A$ by Lemma 1. Also $\theta_p^{*-1}(B)$ is prime by Lemma VIII. Hence $\theta_p^{*-1}(B) = A$ and so $\theta_p^*(A) = B$. Thus $\theta_p^*(A)$ is minimal prime. The last part is proved on similar lines.

THEOREM 2. *If A is a minimal prime ideal (maximal filter) of S/θ_p, then $\theta_p^{*-1}(A)$ is a minimal prime ideal (maximal filter) of S. If S has* 0 *and A is a minimal prime ideal (maximal filter) of S/θ_m, then $\theta_m^{*-1}(A)$ is a minimal prime ideal (maximal filter) of S.*

PROOF. Let A be a minimal prime ideal of S/θ_p. By Lemma VIII, $\theta_p^{*-1}(A)$ is a prime ideal of S. Let B be any prime ideal of S such that $\theta_p^{*-1}(A) \supseteq B$. Then $A \supseteq \theta_p^*(B)$. By Theorem 1 and Lemma 1, it follows that $\theta_p^{*-1}(A) = B$. Hence $\theta_p^{*-1}(A)$ is a minimal prime ideal of S.

Suppose S has 0 and A is a minimal prime ideal of S/θ_m. By Lemma VIII, $\theta_m^{*-1}(A)$ is prime. By Lemma II, $\theta_m^{*-1}(A) \supseteq B$ for some minimal prime ideal B of S. As in the first part, $\theta_m^{*-1}(A) = B$. Thus, $\theta_m^{*-1}(A)$ is minimal prime.

As a consequence of Theorems 1 and 2 and Lemma VIII, we have the following.

THEOREM 3. *There is a bijection between the set of prime ideals (filters) of S and the set of prime ideals (filters) of S/θ_p. Under this bijection, the minimal prime ideals (maximal filters) of S correspond to the minimal prime ideals (maximal filters) of S/θ_p. If S has* 0, *there is a bijection between the set of minimal prime ideals (maximal filters) of S and the set of minimal prime ideals (maximal filters) of S/θ_m.*

THEOREM 4. *If $\{A_i: i \in I\}$ is a family of prime ideals or filters (minimal prime ideals or maximal filters) of S, then*

$$\theta_p^*\left(\bigcap_{i\in I} A_i\right) = \bigcap_{i\in I} \theta_p^*(A_i)\left(\theta_m^*\left(\bigcap_{i\in I} A_i\right) = \bigcap_{i\in I} \theta_m^*(A_i)\right).$$

PROOF. Let $x \in \bigcap_{i\in I}\theta_p^*(A_i)$. Then for each $i \in I$, there exists $x_i \in A_i$ such that $x = \theta_p^*(x_i)$. It follows that $x \in \theta_p^*(\bigcap_{i\in I}A_i)$. Thus $\bigcap_{i\in I}\theta_p^*(A_i) \subseteq \theta_p^*(\bigcap_{i\in I}A_i)$. The reverse inclusion is obvious.

The second part is proved on similar lines.

As a consequence of Lemmas 1 and III and Theorem 4, we have the following.

COROLLARY 1. *If S is intraregular and A is an ideal of S, then $\theta_p^{*-1}\theta_p^*(A) = A$.*

COROLLARY 2. *If S is intraregular and $\{A_i: i \in I\}$ is a family of ideals of S, then $\theta_p^*(\bigcap_{i\in I}A_i) = \bigcap_{i\in I}\theta_p^*(A_i)$.*

The following theorem is easily proved.

THEOREM 5. *For each* $x \in S$, $\theta_p^*(K(x)) = K(\theta_p^*(x))$. *If* $\{A_i: i \in I\}$ *is a family of filters of* S, $\theta_p^*(\bigvee_{i\in I}A_i) = \bigvee_{i\in I}\theta_p^*(A_i)$. *If* $\{B_i: i \in I\}$ *is a family of filters of* S/θ_p, $\theta_p^{*-1}(\bigvee_{i\in I}B_i) = \bigvee_{i\in I}\theta_p^{*-1}(B_i)$.

As an immediate consequence of Theorems 3, 4 and 5, Corollaries 1 and 2 and Lemma VIII, we have the following.

THEOREM 6. *If* S *has* 1, θ_p^* *induces an isomorphism between the lattice of filters of* S *and the lattice of filters of* S/θ_p *which preserves unrestricted joins, unrestricted meets, the property of being a principal filter and the property of being a maximal filter in both directions. If* S *is an intraregular groupoid with* 0, θ_p^* *induces an isomorphism between the lattice of ideals of* S *and the lattice of ideals of* S/θ_p *which preserves unrestricted joins, unrestricted meets, the property of being a principal ideal, the property of being a prime ideal and the property of being a minimal prime ideal in both directions.*

THEOREM 7. *If* S *is an intraregular groupoid with* 0 *and* A *is any ideal of* S, *then* $\theta_m^*(A^*) = \theta_m^*(A)^*$.

PROOF. Let $\{A_i: i \in I\}$ be the family of minimal prime ideals of S not containing A. Then the ideals $\theta_m^*(A_i)$ are precisely the minimal prime ideals of S/θ_m not containing $\theta_m^*(A)$. By Lemma V and Theorem 4 it follows that $\theta_m^*(A^*) = \theta_m^*(A)^*$.

3. **Additional properties of θ_p and θ_m.** As in §2, S will denote a groupoid throughout this section.

THEOREM 8. *Let* θ *be any congruence on* S. *Then* S/θ *is a semilattice if and only if* $\theta_p \subseteq \theta$.

PROOF. The 'If part' is obvious. Suppose S/θ is a semilattice and let $x\theta_p y$. Let θ^* denote the natural homomorphism of S onto S/θ. It is easily seen that $J(\theta^*(x)) = J(\theta^*(y))$. Hence $\theta^*(x) = \theta^*(y)$. Thus $\theta_p \subseteq \theta$.

THEOREM 9. *For any two distinct elements of* S/θ_m, *there exists a minimal prime ideal of* S/θ_m *containing exactly one of them.*

PROOF. Let $\theta_m^*(x), \theta_m^*(y)$ be any two distinct elements of S/θ_m so that $x, y \in S$. Then there exists a minimal prime ideal A of S containing exactly one of x, y, say x. Clearly, $\theta_m^*(A)$ is a minimal prime ideal of S/θ_m such that $\theta_m^*(x) \in \theta_m^*(A)$ and $\theta_m^*(y) \not\in \theta_m^*(A)$.

As a consequence of Theorem 9 and Lemmas IV and VI, we have the following.

THEOREM 10. *If* S *is an intraregular groupoid with* 0, *then* S/θ_m *is a disjunction semilattice.*

THEOREM 11. *Let* S *be an intraregular groupoid closed for pseudocomplements. Then* $x\theta_m y \Leftrightarrow x^* = y^*$ *and* S/θ_m *is a Boolean algebra.*

PROOF. By Lemma IV it follows that $x\theta_m y \Leftrightarrow x^* = y^*$. It is easily seen that

$\theta_m^*(x^*)$ is the pseudocomplement of $\theta_m^*(x)$ for each $x \in S$. By Theorem 10 and Lemma VII it follows that S/θ_m is a Boolean algebra.

4. **Two congruences on a lattice**. Throughout this section L will denote a lattice.

For $x, y \in L$ define $x\phi_p y$ ($x\phi_m y$) to mean that every prime lattice-ideal (minimal prime lattice-ideal) containing x contains y and vice versa. Then it is easily seen that ϕ_p and ϕ_m are congruences on L and that L/ϕ_p and L/ϕ_m are distributive lattices.

Analogues of many of the results in §§2 and 3 hold for ϕ_p and ϕ_m. For example, we have the following.

THEOREM 12. *Let θ be any congruence on L. Then L/θ is a distributive lattice if and only if $\phi_p \subseteq \theta$.*

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