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MORE ON GROUPS IN WHICH EACH ELEMENT COMMUTES WITH ITS ENDOMORPHIC IMAGES

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ABSTRACT. R. Faudree has given examples of nonabelian groups which have the property cited in the title. His groups are p-groups such that (i) $Z(G) = G' = G^p = U(G)$, (ii) each endomorphism ϕ (which is not an automorphism) has $(G)\phi < Z(G)$, and (iii) each automorphism is central. In this paper the necessity of these conditions is explored. It is also shown that, for $p = 2$, Faudree's example does not in fact have the property cited in the title.

1. **Introduction**. In [3], R. Faudree gave counterexamples to the conjecture that a group in which each element commutes with its endomorphic images must be abelian. For each prime p he defined a group by:

$$G = \langle a_1, a_2, a_3, a_4: a_i^{p^2} = 1; (a_i, a_j, a_k) = e, (1 \leqslant i, j, k \leqslant 4);$$
$$\text{and } (a_1, a_2) = a_1^p, (a_1, a_3) = a_3^p, (a_1, a_4) = a_4^p,$$
$$(a_2, a_3) = a_2^p, (a_2, a_4) = e, (a_3, a_4) = a_3^p\rangle.$$

This group has exponent p^2 and each element of G has a unique representation in the form

$$a_1^{k_1}a_2^{k_2}a_3^{k_3}a_4^{k_4}, \qquad 0 \leqslant k_i < p^2, 1 \leqslant i \leqslant 4,$$

so that $|G| = p^8$. That these groups have the property that each element commutes with its endomorphic images is demonstrated by proving that

(i) $G^p = U(G) = G' = Z(G)$, where $U(G) = \{g \in G: |g| \text{ divides } p\}$,

(ii) each strict endomorphism (i.e. an endomorphism that is not an automorphism) of G has its image in $Z(G)$, and

(iii) each automorphism of G is central.

It is obvious that (ii) and (iii) imply that G has the desired property. A group in which each element commutes with its endomorphic images will be called an "E-group".

Conditions (i), (ii), and (iii) are very strong conditions. In §2 we study the

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necessity of these conditions for certain E-groups and in §3 some motivation for studying E-groups is given.

For future use, we present the following definition from [5].

DEFINITION 1. An ordered set of elements $a_1, a_2, \ldots, a_n$ of a group G of orders $y_1, y_2, \ldots, y_n$, respectively, is called a uniqueness basis of G if each element g of G can be expressed uniquely in the form $g = a_1^{x_1} \cdots a_n^{x_n}$, $0 \leqslant x_i < y_i$.

2. **Necessary conditions for some E-groups.** Faudree's groups are the only published examples of nonabelian E-groups. As Faudree did, we investigate a finite nonabelian pE-group (a group which is both a p-group and an E-group) of exponent p^2 and without a direct factor to see if the special conditions of Faudree's groups are necessary conditions for E-groups.

The following theorem is not new. It was given to the author in 1969 by both B. H. Neumann and M. Suzuki. Note that the theorem guarantees that elements of order p of a finite pE-group are in the center.

THEOREM 1. *Let G be a finite pE-group. If G/G' has exponent p^r, then all elements of order $\leqslant p^r$ are in $Z(G)$.*

PROOF. Let $g \in G$ have order $\leqslant p^r$ and let h be an element of G such that hG' has order p^r in G/G'. Then, by the standard result on expressing a finite abelian group as a direct product of cyclic groups, $\langle hG' \rangle$ is a direct factor of G/G'. Hence there exists a homomorphism of G onto a cyclic group $\langle x \rangle$ of order p^r which sends h to x. Since there exists a homomorphism of $\langle x \rangle$ onto $\langle g \rangle$ sending x to g, there is an endomorphism of G which sends h to g. Thus g commutes with h.

If $a \in G$ and $a \notin \langle h \rangle$, then $hG' \cdot aG' = h_1 G'$ where $|h_1 G'|$ is p^r in G/G'. Then $a = h^{-1} h_1 g'$ where $g' \in G'$. Since G is nilpotent, G' is contained in the Frattini subgroup $\Phi(G)$ and it follows that $H = \{h \in G\colon |hG'| = p^r \text{ in } G/G'\}$ is a generating set for G. Since g commutes with each element of H, $g \in Z(G)$.

COROLLARY 1.1. *Let G be a finite p-group. If an element of maximal order in G retains its order in G/G', then G cannot be a nonabelian E-group.*

The material presented in §3 will show that the corollary is equivalent to Lemma 3.2 of [8].

We now investigate the necessity of condition (i) and also show that the Frattini subgroup $\Phi(G)$ is equal to G'.

THEOREM 2. *Let G be a finite nonabelian pE-group of exponent p^2 and without a direct factor. Then*

$$G^p = U(G) = G' = \Phi(G) \leqslant Z(G)$$

and G is nilpotent of class 2.

PROOF. By Theorem 1, $U(G) \leqslant Z(G)$. Obviously, $G^p \leqslant U(G)$. Also

$$(a, b)^p = \left(a^{-1}(b^{-1}ab)\right)^p = a^{-p}(b^{-1}ab)^p = a^{-p}b^{-1}a^pb = e$$

so that $G' \leqslant U(G)$. The second equality is justified by the fact that if an inner automorphic image of an element commutes with the element, it also commutes with the inverse of the element.

If $g \in G$ is such that $|g| = p$ and $g \not\in \Phi(G)$, then there exists some maximal subgroup H such that $g \not\in H$. Then since $g \in Z$, $G = H \odot \langle g \rangle$ which is a contradiction. Hence $U(G) \leqslant \Phi(G)$. For any finite p-group $\Phi(G) = G'G^p$ (see Theorem 5.4.5 of [1]) and, by Corollary 1.1, if $|a| = p^2$ then $a^p \in G'$. Thus $G^p \leqslant G'$ and $\Phi(G) = G'$.

Since $G' \leqslant U(G) \leqslant \Phi(G)$ and $U(G) \leqslant Z(G)$, we obtain

$$U(G) = G' = \Phi(G) \leqslant Z(G).$$

This also tells us that G is nilpotent of class 2.

We interrupt the proof of Theorem 2 to present

LEMMA 1. *If G is as in Theorem 2 and p is odd, then G is p-abelian.*

PROOF. Since $G' \leqslant Z(G)$, $(ab)^n = a^nb^n(b, a)^{n(n-1)/2}$ for all $a, b \in G$ and n a positive integer. See, for example, page 191 of [4]. Let $n = p$ and note that $(b, a)^p = e$ to obtain $(ab)^p = a^pb^p$.

PROOF (THEOREM 2). Assume p is odd. Since G is p-abelian it is also regular. P. Hall in (4.53) of [5] has shown that each regular p-group has a uniqueness basis (ub). Let $a_1, \ldots, a_n$ be a ub for G. Since the exponent of G is p^2, it follows from page 91 of [5] that $|a_i| = p^2$ for each i. (It might be noted that Faudree's four generators constitute a ub.) For $a \in G$ such that $|a| = p$ write $a = a_1^{x_1} \cdots a_n^{x_n}$. Then $e = a^p = a_1^{x_1p} \cdots a_n^{x_np}$. But, by the definition of ub, $a_i^{x_ip} = a_i^{p^2} = e$ and p must divide x_i for each i. If x_i is written as pr_i, then $a = (a_1^{r_1} \cdots a_n^{r_n})^p$ so that $a \in G^p$ and $U(G) \leqslant G^p$ and, in fact, $U(G) = G^p$. Note that we have proved that each element of order p is a pth power if p is odd.

If $p = 2$ and G^p is, as usual, the subgroup generated by the pth powers we have that $G^p \lhd G$ since $G^p \leqslant U(G) \leqslant Z(G)$. But then G/G^p has exponent 2 and so is abelian. Thus $G' \leqslant G^p$ so that $G^p = U(G) = G'$ and the proof of Theorem 2 is complete.

THEOREM 3. *If G is as in Theorem 2 with p odd and if a ub for G has four elements, then $U(G) = Z(G)$.*

PROOF. Let $g \in G$ have order p^2 and be in $Z(G)$. Then g may be chosen as a member of a ub. Since the ub, which is a generating set, has four elements, G' is generated by the commutators determined by the six possible distinct pairs of the four generators. However, the three commutators involving g will each be equal to e. Thus G' will be generated by three elements and will have order at most p^3. But the ub guarantees that $U(G)$ has order p^4 as it is generated by the pth powers of the four generators. Thus Theorem 1 is contradicted and so no such g can exist.

J. D. P. Meldrum has recently informed the author that he has constructed a nonabelian pE-group, p odd, of exponent p^2 and with no direct factors having five generators and with the property that $U(G) < Z(G)$. All told, we see that condition (i) had to be observed by Faudree since he worked with four generators, but that $Z(G)$ may contain elements of order p^2 if more generators than four are involved.

We now show that for $p = 2$, the group given by Faudree is not an E-group. From the defining relations it follows that $(a_1a_2a_3)^2 = e$. However $a_1(a_1a_2a_3) = a_1^2a_2a_3$ whereas $(a_1a_2a_3)a_1 = a_2a_3^3$. Thus, if $|G| = p^8$, $a_1a_2a_3$ is an element of order $p = 2$ which is not in $Z(G)$ and Theorem 1 is contradicted. If $|G| \neq p^8$, there is the possibility that $a_1^2a_2a_3 = a_2a_3^3$. However this would mean that $a_1^2 = a_3^2$. In turn this would imply that $a_1a_3a_4$ has order 2. But $a_1(a_1a_3a_4)$ is equal to $(a_1a_3a_4)a_1$ only if $a_1^2 = a_4^2$. Returning to $a_1a_2a_3$, it is seen that $a_3(a_1a_2a_3) = (a_1a_2a_3)a_3$ only if $a_1^2 = a_2^2$. Thus, to avoid a contradiction to Theorem 1, we must have $a_1^2 = a_2^2 = a_3^2 = a_4^2$ and $|G'| = 2$. By Theorem 2, $G' = U(G)$ and so G has a unique subgroup of order 2. But then G must be a (generalized) quaternion group (see page 189 of [4]). However the work in [6] shows that no quaternion group is an E-group.

We now turn attention to condition (ii).

THEOREM 4. *An endomorphic image of an E-group is an E-group.*

PROOF. Let G be an E-group, H the image of G under the endomorphism γ, and α an endomorphism of H. Then $\gamma\alpha$ is an endomorphism of G whose image is $H\alpha$. Let β be an endomorphism of H and $h \in H$. Select $g \in G$ such that $g\gamma = h$. Then

$$g(\gamma\alpha)\cdot g(\gamma\beta) = g(\gamma\beta)\cdot g(\gamma\alpha)$$

(the commutativity of these elements follows from the definition of an E-group; see Theorem 6) and this implies that

$$h\alpha \cdot h\beta = h\beta \cdot h\alpha.$$

Thus any two endomorphic images of h commute and the group H is an E-group.

COROLLARY 4.1. *If G is a finite E-group which has no nonabelian E-group as a subgroup, the proper endomorphic images of G must be abelian, i.e. G' is in the kernel of each strict endomorphism of G.*

COROLLARY 4.2. *If G is a finite nonabelian E-group of exponent p^2 which has no nonabelian E-group as a subgroup, then $U(G)$ is in the kernel of each strict endomorphism of G. Thus the exponent of the range is p and a strict endomorphism has its image in $Z(G)$.*

From the last corollary we see that condition (ii) is necessary in some relatively uncomplicated E-groups of exponent p^2 and can guess that there exist E-groups of exponent p^2 in which condition (ii) does not hold.

The author has found no evidence that condition (iii) is necessary for E-groups of exponent p^2. However, the following theorem does give some information about central automorphisms.

THEOREM 5. *Let G be an E-group of exponent* p^2. *An automorphism of G is central if and only if it fixes each element of* G^p.

PROOF. Let α be an automorphism of G which fixes each element of G^p and write $g\alpha$ as ga, $a \in G$. Then $(g\alpha)^p = (ga)^p$ and $g^p\alpha = g^pa^p$ since g and a commute because g and ga do. But then $g^p = g^pa^p$ and $a^p = e$. Thus $a \in U(G) \leqslant Z(G)$ and α is central. Conversely, it is easily seen that, in any group, a central automorphism fixes G' elementwise. In our case $G^p = G'$ and the result follows.

3. **Background.** The near ring generated additively by the endomorphisms (inner automorphisms) of a (not necessarily commutative) group $(G, +)$ is designated by $E(G)$ $(I(G))$. These morphism near rings are, in fact, distributively generated (d.g.) near rings with addition of functions being pointwise and multiplication being composition. For definitions of "near ring" and "d.g.", and an introduction to the concept of the endomorphism near ring $E(G)$, see [7].

The question as to when $I(G)$ is a ring (sometimes rephrased to "when G is an I-group") has been settled. A recent formulation of the answer is given by A. J. Chandy in [2].

An I-group is nilpotent of class at most 3. All groups of class at most 2 are I-groups. If an I-group contains no elements of order three, it is of class at most 2. In view of this nilpotence and the easily established fact that a direct summand of an E-group is an E-group, one studies p-groups without direct summands in starting to investigate finite groups G for which $E(G)$ is a ring.

That the use of the expression "E-group" in this section is consistent with the use of the term in §2 is shown in the following theorem. This proof is similar to an unpublished proof given by A. J. Chandy. The theorem itself is part of the folklore.

THEOREM 6. *In a group* $(G, +)$ *an element g commutes with its endomorphic images if and only if any two endomorphic images of G commute.*

PROOF. Assume an element commutes with its endomorphic images and let α and β be endomorphisms of G and i the identity map on G. Then

$$\begin{aligned}
\alpha + \beta &= \alpha + \beta + \alpha\beta + i - i - \alpha\beta \\
&= (i + \alpha) + (i + \alpha)\beta - i - \alpha\beta \\
&= (i + \alpha)(i + \beta) - i - \alpha\beta \\
&= (i + \alpha)(\beta + i) - i - \alpha\beta \\
&= (i + \alpha)\beta + (i + \alpha)i - i - \alpha\beta \\
&= \beta + \alpha\beta + i + \alpha - i - \alpha\beta \\
&= \beta + \alpha(\beta + i) - \alpha\beta \\
&= \beta + \alpha(i + \beta) - \alpha\beta \\
&= \beta + \alpha + \alpha\beta - \alpha\beta \\
&= \beta + \alpha.
\end{aligned}$$

The converse is immediate since i is an endomorphism.

This theorem and the result from near ring theory that $(E(G), +)$ is abelian if and only if multiplication is left and right distributive over addition tells us that Faudree's groups, for odd p, are nonabelian groups for which $E(G)$ is a ring. The question of when $E(G)$ is a ring is treated in Maxson's paper [8] which has already been cited and by B. C. McQuarrie and J. J. Malone in [9]. However, no examples of E-groups are given in those papers.

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