PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 65, Number 2, August 1977

ON THE DIAMETERS OF COMPACT RIEMANN SURFACES

FWU-RANQ CHANG[1]

ABSTRACT. We derive an inequality relating the diameter and the length of a simple closed geodesic on a compact Riemann surface.

1. **Introduction.** Let S be a compact Riemann surface of genus $g \geqslant 2$. Let G be the Fushsian group representing S. The metric on $S = U/G$ is the Poincaré metric, which is induced from the Poincaré metric $|dz|/y$ in the upper half plane U. This is the only metric we use throughout this paper. Let α be a simple closed geodesic on S. If we vary the conformal structure on S so that the length of α goes to zero, then the surface S goes to the boundary of Teichmüller space. This kind of deformation was justified by Keen [2] using the existence of collars on Riemann surfaces.

In [4] Mumford proved a general compactness theorem for Fuchsian groups of the first kind under the hypotheses that all groups G considered are torsion free and U/G is compact. These additional conditions were removed by Bers [1]. Along the lines of Mumford's proof, he derived an inequality relating the diameter and a shortest simple closed geodesic on a compact Riemannian manifold. In Riemann surface theory it can be read as follows. Let S be a compact Riemann surface of genus g. Let $\mathbf{d}$ be the diameter of S and let $\mathbf{m}$ be the length of a shortest simple closed (nontrivial) geodesic on S; then

$$\mathbf{md} \leqslant 2 \operatorname{area}(S).$$

In this paper, we shall find a sharper inequality and an inequality in the reverse direction, from which we conclude that if S goes to the boundary of Teichmüller space, then the diameter of S will go to infinity. As a matter of fact, the inequality we found is true for any simple closed geodesic (not necessarily of shortest length). Moreover, we allow the group G representing S to have elliptic elements.

The author wishes to thank Professor Bernard Maskit for many helpful suggestions.

2. **Definitions and statements of results.** Let G be a finitely generated Fuchsian group of the first kind with signature $\sigma = (g, n;\ \nu_1, \nu_2, \ldots, \nu_n)$, where g and n are positive integers or zero, the ν_i are integers or the symbol

Received by the editors May 18, 1976 and, in revised form, November 11, 1976.

AMS (MOS) subject classifications (1970). Primary 30A46.

[1]This paper is based on a part of the author's Ph. D. dissertation written under Professor Bernard Maskit at the State University of New York, Stony Brook, 1976.

∞, and $2 \leqslant \nu_1 \leqslant \nu_2 \leqslant \cdots \leqslant \nu_n \leqslant \infty$. It is known that the area of G is given by

$$\mathbf{A} = \operatorname{area}(G) = \iint_{U/G} y^{-2}\, dx\, dy$$
$$= 2\pi\left(2g - 2 + n - \nu_1^{-1} - \cdots - \nu_1^{-1}\right) > 0.$$

Note that U/G is compact if and only if $n = 0$ or $\nu_n < \infty$; we call such signature of compact type. We denote by $\operatorname{diam}(G)$ the diameter of U/G. The diameter is finite if and only if U/G is compact.

We denote by $X(\sigma)$ the set of conjugacy classes $[G]$ of Fuchsian groups G with signature σ. Mumford's compactness theorem states that if σ is of compact type, then the subset of $X(\sigma)$ corresponding to groups G so that all geodesics on U/G have length $\geqslant c$ (a constant) is compact.

Let $S = U/G$ be a compact Riemann surface. Let α be a minimal geodesic of length $\mathbf{a}$ on S. By a band B around α of radius $\mathbf{b}$ we mean the union of all geodesics on S perpendicular to α, where each is of length $\mathbf{b}$. Since the Poincaré metric on U is given by

$$\lambda(z)|dz| = 2(z - \bar{z})^{-1}|dz|,$$

the ray $\theta = \theta_0$, $0 < \theta_0 < \pi/2$, is of distance $\log|\csc\theta_0 + \cot\theta_0|$ to the imaginary axis $\theta = \pi/2$. Lifting B up to the upper half plane U with α lying on the imaginary axis, an easy computation shows that B is a region bounded by the curves $\rho = 1$, $\rho = e^{\mathbf{a}}$, $\theta = \csc^{-1}(\cosh(\mathbf{b}))$ and $\theta = \pi - \csc^{-1}(\cosh(\mathbf{b}))$. One easily sees that the noneuclidean area of B is $2\mathbf{a}\sinh(\mathbf{b})$.

Now we state the main results.

THEOREM. *Let G be a Fuchsian group so that U/G is compact with* $\mathbf{d} = \operatorname{diam}(G)$ *and* $\mathbf{A} = \operatorname{area}(G)$. *Then we have*

(1) $$2\mathbf{r}\sinh(\mathbf{d}) \geqslant \mathbf{A},$$

where $\mathbf{r}$ is the length of a simple closed geodesic on S; and

(2) $$2\sinh(\mathbf{m}/4)\,\mathbf{d} \leqslant \mathbf{A},$$

where $\mathbf{m}$ is the length of a shortest simple closed geodesic on S.

The following corollaries follow from the previous theorem together with Mumford's compactness theorem.

COROLLARY 1. *Let σ be the type $(g, 0)$. If G is on the boundary of Teichmüller space $X(\sigma)$, then U/G is not compact.*

With Bers' generalization of Mumford's theorem, we have

COROLLARY 2 (BERS [1]). *Let σ be of compact type. The subset of $X(\sigma)$ corresponding to groups G with* $\operatorname{diam}(G) \leqslant c < \infty$ *is compact.*

3. **Proof of the theorem.** Let β be a simple closed geodesic on $S = U/G$ with length $\mathbf{r}$. Let p be any point on β which is not a fixed point of G. Form

the Dirichlet region D of G in U with center at $p = ie^{\mathbf{r}/2}$ and β lying on the imaginary axis of U. Then D is contained in a strip bounded by the curves $\rho = 1$ and $\rho = e^{\mathbf{r}}$. We construct a band B around β of radius $\mathbf{d}$. Lifting B to U, B is the region bounded by the curves $\rho = 1$, $\rho = e^{\mathbf{r}}, \theta = \csc^{-1}(\cosh(\mathbf{d}))$ and $\theta = \pi - \csc^{-1}(\cosh(\mathbf{d}))$. Since any point q on S is of distance at most $\mathbf{d}$ to the center $p = ie^{\mathbf{r}/2}$, q is of distance at most $\mathbf{d}$ to β. Thus $D \subset B$ and hence,

$$\mathbf{A} = \operatorname{area}(G) = \operatorname{area}(D) \leqslant \operatorname{area}(B) = 2\mathbf{r}\sinh(\mathbf{d}).$$

This proves (1).

Let δ be a minimal geodesic realizing the diameter of S with endpoints x and y. Let B' be the band around δ of radius $\mathbf{m}/4$. We first prove that no two such geodesics of B' meet. Suppose δ_1 and δ_2 meet at the point w. Let z_1, z_2 be the points on δ from which δ_1, δ_2 originate, and $\mathbf{e}$ be the distance from z_1 to z_2 along δ. Then we can go from x to y by going from x to z_1 on δ, following δ_1, then δ_2 and going from z_2 to y on δ. This has length $\leqslant \mathbf{d} - \mathbf{e} + \mathbf{m}/2$, and since δ is the shortest path from x to y, $\mathbf{d} \leqslant \mathbf{d} - \mathbf{e} + \mathbf{m}/2$, i.e., $\mathbf{e} \leqslant \mathbf{m}/2$. But then δ_1, δ_2 and the part of δ between z_1 and z_2 is a closed curve τ of length at most $\mathbf{m}$. τ is certainly not homotopic to zero, for otherwise τ would be lifted to a triangle in the upper half plane U with two right interior angles. Moreover, τ has corners and so it is not a geodesic. Therefore there is a closed geodesic freely homotopic to τ of length $< \mathbf{m}$, which is impossible.

This shows that, with the choice of the radius $\mathbf{m}/4$, the whole band B' is contained in S. So we have

$$2\sinh(\mathbf{m}/4)\mathbf{d} = \operatorname{area}(B') \leqslant \operatorname{area}(S) = \mathbf{A},$$

and the theorem is proved.

References

1. L. Bers, *A remark on Mumford's compactness theorem*, Israel J. Math. **12** (1972), 400–407.

2. L. Keen, *Collars on Riemann surfaces*, Ann. of Math. Studies, No. 79, Princeton Univ. Press, Princeton, N. J., 1974, pp. 263–268.

3. J. Lehner, *Discontinuous groups and automorphic functions*, Math. Surveys, no. 8, Amer. Math. Soc., Providence, R. I., 1964.

4. D. Mumford, *A remark on Mahler's compactness theorem*, Proc. Amer. Math. Soc. **28** (1971), 289–294.

Department of Mathematics, State University of New York, Stony Brook, New York 11794

Current address: Department of Economics, University of Chicago, Chicago, Illinois 60637