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THE NONFINITENESS OF Nil

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ABSTRACT. We show that Nil R is finitely generated only when it vanishes.

Bass and Murthy [2] gave the first examples of "nice" groups G such that Wh G is not finitely generated. Their examples result from calculating Nil R for certain rings R. (See [1, Chapter 12] for the basic facts about Nil R.) We show this is a general "pathology" for Nil R. Let R be any ring with unit 1.

THEOREM. *If* Nil $R \neq 0$, *then* Nil R *is not finitely generated.*

Our proof is based on three lemmas which we now discuss. Let n be a positive integer, t an indeterminate, $R[t]$ and $R[t^n]$ polynomial rings, and σ: $R[t^n] \to R[t]$ the canonical inclusion. Recall that σ induces induction and restriction (transfer) maps

$$\sigma_*: K_1R[t^n] \to K_1R[t],$$
$$\sigma^*: K_1R[t] \to K_1R[t^n],$$

respectively. The following is immediate.

LEMMA 1. *The composite $\sigma^*\sigma_*$ is multiplication by n on $K_1R[t^n]$.*

Next, we recall how Nil R embeds (as a direct summand) in $K_1R[t]$ and in $K_1R[t^n]$. If the nilpotent matrix N represents an element of Nil R then $I + Nt$ represents the corresponding element of $K_1R[t]$ where I is the identity matrix. Denote this map by α and use α' for the map Nil $R \to K_1R[t^n]$ induced by $N \to I + Nt^n$.

LEMMA 2. *The image of α' is mapped into the image of α by σ_*.*

PROOF. Let N represent some $x \in$ Nil R, then $I + Nt^n$ represents $\sigma_*\alpha'(x) \in K_1R[t]$. It is well known that the image of α is precisely the kernel of ε_*: $K_1R[t] \to K_1R$ where ε: $R[t] \to R$ is the evaluation homomorphism $p(t) \to p(0)$ for $p(t) \in R[t]$; clearly, the class of $I + Nt^n$ is in the kernel of ε_*.

LEMMA 3. *For each $x \in$ Nil R, there exists an integer $K(x)$ such that $\sigma^*\alpha(x) = 0$ for all integers $n \geqslant K(x)$.*

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PROOF. Represent x by an $s \times s$ nilpotent matrix N, then the $ns \times ns$ matrix M (described below in blocked form) represents $\sigma^*\alpha(x)$

$$M = \begin{bmatrix} I & & & Nt^n \\ N & I & & \\ & \ddots & \ddots & \\ & & N & I \end{bmatrix};$$

namely, M has I down the diagonal, N down the first subdiagonal, Nt^n in the upper right corner, and the 0 matrix elsewhere. Let $K(x)$ be the degree of nilpotency of N; i.e., $N^n = 0$ for all $n \geqslant K(x)$. Let A be the $ns \times ns$ matrix whose blocked description is the same as M except Nt^n is replaced by 0; i.e.,

$$A = \begin{bmatrix} I & & & \\ N & I & & \\ & \ddots & \ddots & \\ & & N & I \end{bmatrix}.$$

One sees (by an easy calculation) that $A^{-1}M = I + T$ where T is a *strictly* upper triangular matrix provided $n \geqslant K(x)$. Since A and $I + T$ clearly represent zero in $K_1R[t^n]$, so does M.

PROOF OF THEOREM. We proceed by contradiction; i.e., assume Nil R is nonzero but finitely generated. Hence there is a prime p such that multiplication by p is a monomorphism of Nil R. In particular, Lemma 1 together with Lemma 2 imply that $\sigma^*\alpha$: Nil $R \to K_1R[t^n]$ is nonzero when $n = p^i$ for all $i > 0$. But this contradicts Lemma 3 since we assumed Nil R is finitely generated.

REMARKS. There are other Nil-type groups in algebraic K and L-theory. They arise geometrically when studying codimension-1 submanifolds. It seems plausible to conjecture the analogue of our Theorem for all these groups. All examples known to me of nonfinite generation for K and L functors applied to "nice" rings arise from these Nil-type groups. For other particular instances of this, see the following articles: [3] and [4] for UNil, [6] for Nil_2, and [5] for $\mathrm{Wh}_1(\; ; \;)$. This phenomena is a major stumbling block to understanding nonsimply connected manifolds.

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