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AN EXAMPLE OF A SPACE WHICH IS COUNTABLY COMPACT WHOSE SQUARE IS COUNTABLY PARACOMPACT BUT NOT COUNTABLY COMPACT

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ABSTRACT. A subspace P of $\beta N - N$ is obtained whose square is disjoint from the graph, G, of a pre-selected homeomorphism $f: \beta N \to \beta N$ that has no fixed points. The construction is performed in such a way that, for $X = P \cup N$, all countable subsets of $X^2 - G$ will have a limit point in X^2. We use the following lemma: If $K \subset (\beta N)^2 - G$ is countably infinite, then $|\mathrm{cl}_{(\beta N)^2} K - G| = 2^c$.

We construct the space X using the technique of J. Novák [N]. In the reference cited, Novák constructs a countably compact space whose square is not countably compact. Several versions have appeared in the literature. H. Terasaka's example [T] is presented in Gillman and Jerison, *Rings of continuous functions* [GJ] and in Steen and Seebach, *Counterexamples in topology* [SS]. Novák's example was modified by Frolík in [F]. The latter version is presented by Engelking in [E], *Outline of general topology*.

A subspace P of $\beta N - N$ will be obtained whose square is disjoint from the graph, G, of a preselected homeomorphism f: $\beta N \to \beta N$ that has no fixed point, but has the property that $f^2 = f$. The notation of [GJ] is used, primarily. The construction will be performed in such a way that all countable subsets of $X^2 - G$ will have a limit point in X^2, where $X = P \cup N$. Then X will be countably compact since it is homeomorphic to a closed subset of $X^2 - G$. Moreover, $G \cap X^2 = \{(n, f(n)): n \in N\}$ is closed in X^2 and is an infinite discrete set, so X^2 is not countably compact. But $X^2 = (X^2 \cap G) \cup (X^2 - G)$ is the disjoint union of a countably compact subspace and a countable, clopen discrete subspace and hence is countably paracompact.

The burden of proof is borne mostly by the following

LEMMA. *If $K \subset (\beta N)^2 - G$ is countably infinite then* $|\mathrm{cl}_{(\beta N)^2} K - G| = 2^{2^\omega}$.

PROOF. Suppose that $K \subset (\beta N)^2 - G$ is countably infinite. We let π_1 and π_2 denote the projections onto the first and second factors of subsets of $(\beta N)^2$. If there is a point $p \in N$ such that $H = K \cap (\{p\} \times N)$ is infinite then $|\mathrm{cl}\, H| \geq |\mathrm{cl}(\pi_2 H)| = 2^c$, noting that $\mathrm{cl}\, \pi_2 H = \pi_2 \mathrm{cl}\, H$. But $|G \cap (\{p\} \times$

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$\beta N)| = 1$ so that the lemma holds in this and the analogous case, H-$(\beta N \times \{p\}) \cap K$.

We may now assume throughout (by choosing an appropriate infinite subset of K) that

$$(*) \qquad |K \cap (\{p\} \times \beta N)| \vee |K \cap (\beta N \times \{p\})| \leq 1 \quad \text{for all } p \in \beta N, \text{ and } \pi_i K \text{ is infinite}, \quad i = 1, 2.$$

If $K' \subset K$ is countably infinite and has the property that $\mathrm{cl}(f[\pi_1[K']]) \cap \mathrm{cl}(\pi_2[K']) = \varnothing$, then it is easy to establish that $|\mathrm{cl}_{(\beta N)^2} K'| = 2^{2^\omega}$ and that $\mathrm{cl}_{(\beta N)^2} K' \cap G = \varnothing$, from which the lemma follows. We now devote our attention to producing such a subset of K.

Every countably infinite subset of βN has a countably infinite subset whose topology inherited from βN is discrete. Now, using this fact, choose an infinite subset $K^* \subset K$ such that $\pi_1[K^*]$ is discrete. Then $f[\pi_1[K^*]]$ is discrete, since f is a homeomorphism. Apply this same technique again to obtain $K^{**} \subset K^*$, countably infinite, such that $\pi_2[K^{**}]$ is discrete. By assumption $(*)$, K^{**} has the property that $f[\pi_1[K^{**}]]$ and $\pi_2[K^{**}]$ are infinite, discrete topological subspaces of βN. Since it is a bit tedious to carry the **'s about, let us assume without loss of generality that K has the latter property to begin with.

Now cull K again. Let K be enumerated as $\{(p_1, q_1), (p_2, q_2), \ldots\}$. Let $i_1 = 1$. Let U_1 be a neighborhood of q_{i_1} which misses $f(p_{i_1})$ and infinitely many points of $f[\pi_1[K]]$ and whose intersection with $\pi_2[K]$ is $\{q_{i_1}\}$. Now suppose $i_1, \ldots, i_n$ are selected in such a way that $f[\pi_1[K]] - \bigcup_{i=1}^n U_i$ is infinite and $f(p_{i_j}) \notin \bigcup_{i=1}^n U_n$ for $j = 1, \ldots, n$, and $\bigcup_{i=1}^n U_i \cap \pi_2[K] = \{q_{i_1}, \ldots, q_{i_n}\}$. Now choose i_{n+1} so that $f(p_{i_{n+1}}) \in f[\pi_1[K]] - \bigcup_{i=1}^n U_i$. Then choose U_{n+1} so that, one, it does not contain $f(p_{i_j}), j = 1, \ldots, n+1$; two, its intersection with $\pi_2[K]$ is $\{q_{i_{n+1}}\}$; and three, it misses infinitely many members of $f[\pi_1[K]] - \bigcup_{i=1}^n U_i$. The inductive selection of the sequence $\langle i_1, i_2, \ldots \rangle$ is complete. Denote by K'' the subset $\{(p_{i_1}, q_{i_1}), (p_{i_2}, q_{i_2}), \ldots\}$ of K. Then $\bigcup_{i=1}^\infty U_i$ is a neighborhood of $\pi_2 K''$ and $\bigcup_{i=1}^\infty U_i \cap f[\pi_1[K'']] = \varnothing$. Thus $\mathrm{cl}_{\beta N} f[\pi_1[K'']] \cap \pi_2 K'' = \varnothing$. In an exactly analogous manner, we pick an infinite subset $K' \subset K''$ having the property that $f[\pi_1[K']] \cap \mathrm{cl}(\pi_2[K']) = \varnothing$. Then it follows that $\mathrm{cl}\, f[\pi_1[K']] \cap \mathrm{cl}\, \pi_2[K'] = \varnothing$.

Note that $\mathrm{cl}\, \pi_i K' = \pi_i \mathrm{cl}\, K'$, $i = 1, 2$, so that we actually proved:

If $K \subset (\beta N)^2$ is countably infinite and if $\{p \in \beta N: (\beta N \times \{p\}) \cap K \neq \varnothing\}$ and $\{p \in \beta N: (\{p\} \times \beta N) \cap K \neq \varnothing\}$ are infinite, then

$$|\{r \in \beta N: \exists s \in \beta N, (r, s) \in \mathrm{cl}\, K - G\}| = |\{s \in \beta N: \exists r \in \beta N, (r, s) \in \mathrm{cl}\, K - G\}| = 2^c.$$

Now, beginning the construction of X, we index the countable subsets of $(\beta N)^2 - G$ in type 2^c, $\{K_\beta\}_{\beta < 2^c}$. By the lemma, K_0 has a limit point which is not in $G \cup N^2$, (r_0, s_0). Let $P_0 = \{r_0, s_0\} - N$. Inductively, suppose P_α,

$\alpha < \beta$, are selected so that $P_\alpha \subset P_\gamma$ for $\alpha < \gamma < \beta$ and $f[P_\alpha] \cap P_\alpha = \emptyset$ and $|P_\alpha| = |\alpha|$ if $\alpha \geqslant \omega$ and $|P_\alpha| < \omega$ if $\alpha < \omega$. $|\bigcup_{\alpha<\beta} P_\alpha| = \bigcup_{\alpha<\beta}|\alpha| = |\beta| < 2^{c}$ if $\alpha \geqslant \omega$ and is less than ω if $\alpha < \omega$. Consider K_β. Several cases arise:

(i) $\exists r_\beta$ such that $K_\beta \cap (\{r_\beta\} \times \beta N)$ is infinite.

(a) $r_\beta \in f(\bigcup_{\alpha<\beta} P_\alpha) \subset \beta N - N$. Let $P_\beta = \bigcup_{\alpha<\beta} P_\alpha$. In this case, P will be defined so that $K_\beta \not\subset P$ hence K_β need not have a limit point in X^2.

(b) $r_\beta \notin f[\bigcup_{\alpha<\beta} P_\alpha]$. Choose $s_\beta \in \beta N - (f[\bigcup_{\alpha<\beta} P_\alpha] \cup N)$ so that $(r_\beta, s_\beta) \in \operatorname{cl} K_\beta - (G \cup K_\beta)$. Let $P_\beta = \bigcup_{\alpha<\beta} P_\alpha \cup (\{r_\beta, s_\beta\} - N)$.

(ii) We have an analogous case if $\exists s_\beta$ such that $K_\beta \cap (\beta N \times \{s_\beta\})$ is infinite.

(iii) If no such points exist, apply the lemma, using a simple cardinality argument, to obtain a point (r_β, s_β) so that $r_\beta, s_\beta \notin f[\bigcup_{\alpha<\beta} P_\alpha] \cup N$ and $(r_\beta, s_\beta) \in \operatorname{cl} K_\beta - (G \cup K_\beta)$. Let $P_\beta = \bigcup_{\alpha<\beta} P_\alpha \cup \{r_\beta, s_\beta\}$.

So clearly, $|P_\beta| = |\bigcup_{\alpha<\beta} P_\alpha| = |\beta|$ if $\alpha \geqslant \omega$ and is finite otherwise. Equally clear is that $P_\beta \supset P_\alpha$ for $\alpha < \beta$.

CLAIM. $f[P_\beta] \cap P_\beta = \emptyset$. Let $p \in P_\beta$ and suppose $\exists q \in P_\beta$ such that $f(q) = p$. Note the following:

(i) Obviously, the inductive hypothesis guarantees that not both $p, q \in \bigcup_{\alpha\in\beta} P_\alpha$.

(ii) If $p \in \bigcup_{\alpha<\beta} P_\alpha$ and $q = r_\beta$, we have $f(r_\beta) = p$. So $f(p) = r_\beta$. But r_β was chosen so that $r_\beta \notin f[\bigcup_{\alpha<\beta} P_\alpha]$.

(iii) If $p \in \bigcup_{\alpha<\beta} P_\alpha$ and $q = s_\beta$, $f(s_\beta) = p$, so $f(p) = s_\beta$ and we have a contradiction as above.

(iv) If $p = r_\beta$ and $q = s_\beta$, we have $f(s_\beta) = r_\beta$ so that $f(r_\beta) = s_\beta$. But this gives $(r_\beta, s_\beta) \in G$, a contradiction.

(v) If $p = s_\beta$ and $q = r_\beta$, $f(r_\beta) = s_\beta$, again a contradiction.

The claim now follows.

The inductive construction of the example is now complete. Note that P^2 is countably compact.

REMARKS. (1) The example presented here is a partial negative answer to a question of J. Keesling, whose interest in the problem stems from research announced in [K] concerning hyperspaces. The question, to which I do not know the answer, is: If X is normal and countably compact and X^2 is countably paracompact, is X^2 countably compact? R. G. Woods [Wo] has shown that CH implies that if X is normal, countably compact, extremally disconnected and $|C^*(X)| = 2^\omega$, then X is compact. Thus the present example is not normal assuming CH.

(2) The example presented here also answers in the negative the following question of Morita [M]: If X and Y are countably compact and $X \times Y$ is an M-space, is $X \times Y$ countably compact? The question had been answered in the negative by Steiner [S], assuming the continuum hypothesis. An M-space is the quasi-perfect preimage of a metric space. Note that X^2 is an M-space: It is the free union of a countably compact space and a countably infinite discrete space. See also [Wa, pp. 188–190].

(3) An example, due to Frolík, of countably compact spaces X and Y whose product is pseudocompact but not countably compact is presented by Ginsburg and Saks in [GS]. Only slight modification is needed to yield a countably compact space whose square is pseudocompact but not countably compact. Similar results can be obtained from the example given by Comfort in [C].

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