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ASPHERICAL GENERATORS OF UNORIENTED COBORDISM

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ABSTRACT. A new set of indecomposable generators for MO_* is constructed, each of which is aspherical. Two adjoint maps are used to define these generators. In the process we find a natural basis for $MO_*(\mathbf{Z}_2)$.

1. **Introduction**. Our objective in this paper is to construct a sequence of closed aspherical generators for MO_* as a polynomial ring over $\mathbf{Z}_2$. Recall that the classical definition of an aspherical manifold from knot theory is a closed connected manifold whose universal covering space is contractible; i.e., these are closed connected $K(\pi, 1)$ manifolds. We will also construct a basis of closed aspherical elements for $MO_*(\mathbf{Z}_2)$ as an MO_*-module in §3. However, since the connected sum operation destroys asphericity, it is still unknown if there is a closed connected aspherical representative in each unoriented cobordism class of MO_*. There are many examples of aspherical manifolds in the literature; cf. [2], [5], [7], [8], and [10]. In the case of flat Riemannian manifolds, all of the known examples bound mod 2. This is true even though there are known examples of closed flat Riemannian manifolds with nonzero Stiefel-Whitney classes; cf. [2] and [7].

In all that follows, $I_*(\mathbf{Z}_2)$ will denote the graded unrestricted cobordism ring of smooth manifolds with involution; $MO_*(\mathbf{Z}_2)$, the graded unoriented cobordism ring of smooth manifolds with fixed point free involutions; and MO_*, the graded unoriented Thom cobordism algebra.

2. **The maps** Γ **and** γ. Let $S^1 = \{z \in C \mid |z| = 1\}$. Define the map $\Gamma: I_n(\mathbf{Z}_2) \to I_{n+1}(\mathbf{Z}_2)$ as follows: Take $\{T, M^n\}_2 \in I_n(\mathbf{Z}_2)$ and consider the manifold $S^1 \times M^n$ with two involutions $T_1(z, m) = (-z, T(m))$ and $T_2(z, m) = (\bar{z}, m)$ for $z \in S^1$ and $m \in M^n$. Observe that T_1 is fixed point free, so the quotient of $S^1 \times M^n$ by T_1 is a manifold. T_1 and T_2 commute, so T_2 induces an involution, call it still T_2, on the quotient manifold $(S^1 \times M^n)/T_1$. Put

$$\Gamma(\{T, M^n\}_2) = \{T_2, (S^1 \times M^n)/T_1\}_2 \in I_{n+1}(\mathbf{Z}_2).$$

Similarly, define the map $\gamma: MO_n(\mathbf{Z}_2) \to MO_{n+1}(\mathbf{Z}_2)$ as follows: Take $[\tau, X^n]_2 \in MO_n(\mathbf{Z}_2)$ and consider the manifold $S^1 \times X^n$ with the two fixed point free involutions $T_3(z, x) = (\bar{z}, \tau(x))$ and $T_4(z, x) = (-z, x)$ for $z \in S^1$ and $x \in X^n$. T_3 and T_4 commute, so T_4 induces an involution, call it again T_4, on the quotient manifold $(S^1 \times X^n)/T_3$. This involution is fixed point

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free since T_3 and T_4 have no coincidence. Put

$$\gamma([\tau, X^n]_2) = \left[T_4, (S^1 \times X^n)/T_3\right]_2 \in MO_{n+1}(\mathbf{Z}_2).$$

Observe that our definition of the map Γ differs slightly from the definition in [1]. To resolve this difference let us consider a differentiable involution (T, M^n). Put $\Gamma(\{T, M^n\}_2) = \{T', M'\}_2$, where $M' = (S^1 \times M^n)/T_1$ and T' is the involution induced on M' by T_2. Now, T_1 on $S^1 \times M^n$ is fixed point free. The fixed point set of T_2 on $S^1 \times M^n$ is $\{-1, 1\} \times M^n$. The coincidence of T_1 and T_2 is $\{-i, i\} \times F(T)$, where $F(T)$ is the fixed point set of T on M^n. Thus, the fixed point set of T' on M' is the disjoint union of $F(T)$ and M^n. The normal bundle to this fixed point set is $(\nu \oplus R) \cup R$ where R is the trivial real-line bundle and ν is the normal bundle to $F(T) \subseteq M^n$. Note that the second factor of R appears as the normal bundle to M^n in $S^1 \times M^n$. This is exactly the same fixed point data as the map Γ in [1]. Thus, the two maps agree on the cobordism level.

2.1. DEFINITION. There is a pairing, which is bilinear as an MO_*-module homomorphism, β: $MO_*(\mathbf{Z}_2) \times I_*(\mathbf{Z}_2) \to MO_*$ defined as follows: Take $[\tau, X^n]_2 \in MO_n(\mathbf{Z}_2)$ and $\{T, M^k\}_2 \in I_k(\mathbf{Z}_2)$ and consider the manifold $X^n \times M^k$. Define a fixed point free involution on $X^n \times M^k$ by $T_1(x, m) = (\tau(x), T(m))$. Put

$$\beta([\tau, X^n]_2, \{T, M^k\}_2) = \left[(X^n \times M^k)/T_1\right]_2 \quad \text{in } MO_{n+k}.$$

We have the following relationship between the maps Γ and γ.

2.2. THEOREM. *The maps Γ and γ are adjoint with respect to β.*

$$\beta(\gamma([\tau, X^n]_2), \{T, M^k\}_2) = \beta([\tau, X^n]_2, \Gamma(\{T, M^k\}_2)). \tag{2.3}$$

PROOF. Put

$$Y_1 = \beta(\gamma([\tau, X^n]_2), \{T, M^k\}_2) = \left[\left(M^k \times ((S^1 \times X^n)/T_3)\right)/T'\right]_2$$

and put

$$Y_2 = \beta([\tau, X^n]_2, \Gamma(\{T, M^k\}_2)) = \left[\left(X^n \times ((S^1 \times M^k)/T_1)\right)/T''\right]_2.$$

Denote a point in Y_1 by $[m, [z, x]]$ and a point in Y_2 by $[x, [z, m]]$ for $x \in X^n$, $z \in S^1$, and $m \in M^k$. Define a map f: $Y_1 \to Y_2$ by $f([m, [z, x]]) = [x, [z, m]]$. This map is a diffeomorphism. □

3. **The natural basis** $\{X(k) | k \geqslant 0\}$. We want to consider iterations of the map γ on $[A, S^0]_2 \in MO_0(\mathbf{Z}_2)$. Put

$$X(k) = \gamma^k([A, S^0]_2) \in MO_k(\mathbf{Z}_2) \quad \text{for all } k \geqslant 0.$$

We have the following properties.

3.1. PROPOSITION. $\Delta(X(k+1)) = X(k)$ *for all* $k \geqslant 0$, *where* Δ *is the Smith homomorphism.*

PROOF. We will prove the stronger statement that $\Delta \circ \gamma = \mathrm{id}$. Let $[\tau, M^n]_2$

$\in MO_n(\mathbf{Z}_2)$ and set $\gamma([\tau, M^n]_2) = [\tau', M']_2$. Let $B = \{(z, m) \in S^1 \times M^n | \mathrm{Re}(z) \geqslant 0\}$ and let W be the image of B in M'. Then $W \cup \tau' W = M'$ and $W \cap \tau' W = \partial W = M^n$. By 26.2 of [3], $\Delta([\tau', M']_2) = [\tau', \partial W]_2 = [\tau, M^n]_2$. □

3.2. PROPOSITION. $\varepsilon(X(k)) = 0$ *for all* $k > 0$, *where* ε: $MO_*(\mathbf{Z}_2) \to MO_*$ *is the augmentation map.*

PROOF. We will again prove a stronger statement: $\varepsilon \circ \gamma = 0$. Let $[\tau, M^n]_2 \in MO_n(\mathbf{Z}_2)$. M^n may be regarded as the total space of a line bundle, η, over M^n/τ. By applying γ to $[\tau, M^n]_2$, we add a trivial line bundle to η. Then by applying ε, we pass to the projective bundle $RP(\eta \oplus R)$. Thus,

$$\varepsilon\big(\gamma([\tau, M^n]_2)\big) = \big[RP(\eta \oplus R)\big]_2 \in MO_{n+1}.$$

Now, $\eta \oplus R$ is a two plane bundle over M^n/τ with no second Stiefel-Whitney class. By 2.1 of [4], $[RP(\eta \oplus R)]_2 = 0$. □

3.3. PROPOSITION. $\{X(k) | k \geqslant 0\}$ *forms a homogeneous* MO_**-module basis for* $MO_*(\mathbf{Z}_2)$.

PROOF. It is sufficient to show that $\mu(X(k))$ is a generator of $H_k(RP(\infty); \mathbf{Z}_2)$ for each $k \geqslant 0$, where μ: $MO_*(\mathbf{Z}_2) \to H_*(RP(\infty); \mathbf{Z}_2)$ is the Thom homomorphism. Consider the following commutative diagram

$$\begin{array}{ccc} MO_n(\mathbf{Z}_2) & \xrightarrow{\mu} & H_n(RP(\infty); \mathbf{Z}_2) \\ \Big\downarrow{\scriptstyle \Delta} & & \approx \Big\downarrow{\scriptstyle c\,\cap} \\ MO_{n-1}(\mathbf{Z}_2) & \xrightarrow[\mu]{} & H_{n-1}(RP(\infty); \mathbf{Z}_2) \end{array}$$

where c is the generator of $H^1(RP(\infty); \mathbf{Z}_2)$. Our proof will proceed by induction. Clearly, $\mu(X(0))$ is a generator of $H_0(RP(\infty); \mathbf{Z}_2)$. Assume that $\mu(X(k))$ is a generator of $H_k(RP(\infty); \mathbf{Z}_2)$ for all $k < n$. For n, we have that

$$c \cap \mu(X(n)) = \mu\big(\Delta(X(n))\big) = \mu(X(n-1)),$$

which is a generator of $H_{n-1}(RP(\infty); \mathbf{Z}_2)$. Since capping with c is an isomorphism, $\mu(X(n))$ is a generator of $H_n(RP(\infty); \mathbf{Z}_2)$. □

These three propositions give us the main theorem of this section.

3.4. THEOREM. *The basis* $\{X(k) | k \geqslant 0\}$ *is unique with respect to satisfying* (3.1)–(3.3).

PROOF. Let $\{Y_k | k \geqslant 0\}$ be another homogeneous basis for $MO_*(\mathbf{Z}_2)$ such that (i) $\Delta(Y_{k+1}) = Y_k$ for all $k \geqslant 0$, and (ii) $\varepsilon(Y_k) = 0$ for all $k > 0$. It suffices to show that $Y_k + X(k) = 0$ for all $k \geqslant 0$. Clearly, $Y_0 = X(0)$. Assume that $Y_k + X(k) = 0$ for all $k < n$, and suppose that $Y_n + X(n) \neq 0$. Since $\{X(k) | k \geqslant 0\}$ is a homogeneous basis, we have a unique representation

$$Y_n + X(n) = \sum_{j=0}^{n} a_j X(j), \qquad a_j \in MO_{n-j}.$$

Now,

$$0 = Y_{n-1} + X(n-1) = \Delta(Y_n + X(n))$$
$$= \Delta\left(\sum_{j=0}^{n} a_j X(j)\right) = \sum_{j=1}^{n} a_j X(j-1).$$

Thus, $a_j = 0$ for $1 \leqslant j \leqslant n$ and $Y_n + X(n) = a_0 X(0)$, $a_0 \in MO_n$. Also, $0 = \varepsilon(Y_n + X(n)) = a_0 \varepsilon(X(0))$. But $\varepsilon(X(0)) = 1 \in MO_0$. Thus, $a_0 = 0$ and we have our contradiction. $\square$

With this basis, we have a much more simple representation of any element in $MO_*(\mathbf{Z}_2)$.

3.5. Proposition. *For any* $[\tau, M^n]_2 \in MO_n(\mathbf{Z}_2)$,

$$[\tau, M^n]_2 = \sum_{j=0}^{n} \varepsilon\left(\Delta^j([\tau, M^n]_2)\right) X(j). \tag{3.6}$$

Proof. This follows from repeated applications of Δ and ε. We have that

$$[\tau, M^n]_2 = \sum_{j=0}^{n} a_j X(j), \qquad a_j \in MO_{n-j}.$$

Now,

$$\Delta^j([\tau, M^n]_2) = a_j X(0) + \sum_{i=j+1}^{n} a_i X(i-j) \quad \text{for } 0 \leqslant j \leqslant n.$$

Then, $\varepsilon(\Delta^j([\tau, M^n]_2)) = a_j$. $\square$

3.7. Corollary. $[A, S^n]_2 = \sum_{j=0}^{n} [RP(n-j)]_2 X(j)$.

3.8. *Note*. Since $X(k) = \gamma^k([A, S^0]_2) = \gamma^{k-1}([A, S^1]_2)$, it is clear that $X(k)$ is a closed aspherical manifold for all $k \geqslant 0$.

4. **Aspherical indecomposable generators.** Let us consider $RP(2)$ with the involution T defined in homogeneous coordinates by $T[x_1, x_2, x_3] = [-x_1, x_2, x_3]$. The fixed point set of T is the disjoint union of a point and $RP(1)$.

4.1. Theorem. *$\varepsilon(\Gamma^k(\{T, RP(2)\}_2))$ cobords for k odd and is indecomposable for k even.*

Proof. Recall that $[M]_2$ is indecomposable in MO_* if it cannot be expressed as a sum of products of lower dimensional classes.

If k is odd, then 4.4 of [4] shows that $\varepsilon(\Gamma^k(\{T, RP(2)\}_2))$ cobords. Now, if k is even, let ν denote the normal bundle to the fixed point set of T on $RP(2)$. By 4.2 of [4], $\varepsilon(\Gamma^k(\{T, RP(2)\}_2))$ is indecomposable if and only if $[RP(\nu \oplus (k+1)R)]_2$ is indecomposable. The restriction of ν to $RP(1)$ is the canonical twisted real-line bundle, $\xi \to RP(1)$. Therefore,

$$\left[RP(\nu \oplus (k+1)R)\right]_2 = \left[RP(k+2)\right]_2 + \left[RP(\xi \oplus (k+1)R)\right]_2.$$

By 2.2 of [4], $[RP(\xi \oplus (k+1)R)]_2 = 0$. Since k is even, $[RP(k+2)]_2$ is known to be indecomposable. $\square$

Of course, this theorem only depends on the unrestricted cobordism class of $(T, RP(2))$. Let us consider the 2-torus on which there is an involution with 4 isolated fixed points. Remove an open disk about either one or three of these fixed points. Note that the involution is fixed point free on the boundaries of these disks. Replace these disks with closed Moebius bands by identifying along the boundaries. We want the following involution on each of the Moebius bands. We may consider the Moebius band as the one-cell bundle over $RP(1)$. The involution on the Moebius band will be induced by the antipodal map on the boundary curve. Clearly, the fixed point set of this involution on the Moebius band is $RP(1)$. Let us call the manifold obtained from the 2-torus in this way W^2, and the involution induced on W^2, T'. Then W^2 is aspherical and $\{T', W^2\}_2 = \{T, RP(2)\}_2$. We have the following immediate corollary.

4.2. Corollary. *$\varepsilon(\Gamma^k(\{T', W^2\}_2))$ cobords for k odd and is indecomposable for k even. Furthermore, each of these manifolds is aspherical.*

Proof. The first part is clear from 4.1, and the second part follows from the homotopy exact sequence for a covering space and the fact that W^2 is aspherical. $\square$

So, we have found closed aspherical indecomposable generators for MO_n for n even. However, for n odd we need some further results.

4.3. Definition. Define a map g: $MO_* \to I_*(\mathbf{Z}_2)$ by $g([M^n]_2) = \{S, M^n \times M^n\}_2$, where S is the switching involution $S(x, y) = (y, x)$.

4.4. Lemma. *The map g is a well-defined, additive, and multiplicative homomorphism.*

Proof. To show that g is well defined, it suffices to show that if $[M^n]_2 = 0$, then $g([M^n]_2) = 0$. To compute $g([M^n]_2) = \{S, M^n \times M^n\}_2$, we must look at the normal bundle to $F(S)$, the fixed point set of S. Now, $F(S) \cong M^n$. Let $\nu \to F(S)$ denote the normal bundle to $F(S)$. Now, ν is equivalent to the tangent bundle of M^n, $\tau(M^n)$. Thus, it is sufficient to show that all of the characteristic numbers of $\tau(M^n)$ vanish. By the assumption that $[M^n]_2 = 0$, all of the characteristic numbers of $\tau(M^n)$ are zero.

To show that g is additive, we must show that $\{S, (M \cup N) \times (M \cup N)\}_2 = \{S, M \times M\}_2 + \{S, N \times N\}_2$. Thus, it suffices to show that $\{S, N \times M \cup M \times N\}_2 = 0$. Since $N \times M$ and $M \times N$ are diffeomorphic, it suffices to show that $\{S, (M \times N) \cup (M \times N)\}_2 = 0$. Consider the manifold $M \times N \times I$ with the involution T defined by $T(m, n, t) = (m, n, 1 - t)$. Now, $\partial(M \times N \times I) = (M \times N) \cup (M \times N)$ and $T|\partial(M \times N \times I) = S$. Therefore, $\{S, (M \times N) \cup (N \times M)\}_2 = 0$.

The fact that g is multiplicative is easy to show. $\square$

Let us define decomposable and indecomposable elements in $I_*(\mathbf{Z}_2)$ as in MO_*. Note then that g takes decomposable elements to decomposable elements since g is additive and multiplicative. We need the following two lemmas before proceeding.

4.5. LEMMA. *Let* $\{T, M^n\}_2 \in I_*(\mathbf{Z}_2)$; *then* $\beta([A, S^0]_2, \{T, M^n\}_2) = [M^n]_2 \in MO_*$.

The proof of this lemma is really quite simple and will be left to the reader.

4.6. LEMMA. *If* $\{T, M\}_2 \in I_*(\mathbf{Z}_2)$ *is decomposable, then* $\beta(X(j), \{T, M\}_2)$ *is decomposable.*

PROOF. Since β is bilinear, we may assume that $\{T, M\}_2 = \{\varphi, V\}_2 \cdot \{\psi, W\}_2$. Then

$$\begin{aligned} Y &= \beta(X(j), \{\varphi, V\}_2 \cdot \{\psi, W\}_2) = \beta\big(\gamma^j([A, S^0]_2), \{\varphi, V\}_2 \cdot \{\psi, W\}_2\big) \\ &= \beta\big([A, S^0]_2, \Gamma^j(\{\varphi, V\}_2 \cdot \{\psi, W\}_2)\big) \\ &= \beta\big([A, S^0]_2, \Gamma^{j-1}(\Gamma(\{\varphi, V\}_2) \cdot \{\psi, W\}_2 + \varepsilon(\{\varphi, V\}_2) \cdot \Gamma(\{\psi, W\}_2))\big) \end{aligned}$$

by Theorem 1.3 of [1]. So,

$$\begin{aligned} Y &= \beta\big([A, S^0]_2, \Gamma^{j-1}(\Gamma(\{\varphi, V\}_2) \cdot \{\psi, W\}_2)\big) \\ &\quad + [V]_2 \beta\big([A, S^0]_2, \Gamma^j(\{\psi, W\}_2)\big) \end{aligned} \tag{4.7}$$

from the fact that β is a bilinear, MO_*-map, and from the properties of Γ in [1]. By descent, (4.7) yields

$$\begin{aligned} Y &= \beta\big([A, S^0]_2, \Gamma^j(\{\varphi, V\}_2) \cdot \{\psi, W\}_2\big) + \text{decomposables} \\ &= \big[\Gamma^j(\{\varphi, V\}_2)\big]_2 \cdot [W]_2 + \text{decomposables} \end{aligned}$$

by (4.5). $\square$

Let $n \in \mathbf{Z}^+$ be odd. There are $r, s \in \mathbf{Z}^+$ such that $n + 1 = 2^r(2s + 1)$. Let $j = 2^r - 1$ and $k = s2^r$. Dold has shown [6] that if n is not of the form $2^i - 1$, then

$$[P(j, k)]_2 = [S^j \times CP(k)/(x, z) \sim (-x, \bar{z})]_2$$

is an indecomposable generator of MO_n. From this definition of $P(j, k)$, it is clear that

$$[P(j, k)]_2 = \beta\big([A, S^j]_2, \{C, CP(k)\}_2\big),$$

where C is complex conjugation on $CP(k)$. From 3.7 we have that $[A, S^j]_2 = \sum_{i=0}^{j}[RP(j-i)]_2 X(i)$. So, we now have

$$[P(j, k)]_2 = \sum_{i=0}^{j} [RP(j-i)]_2 \beta\big(X(i), \{C, CP(k)\}_2\big). \tag{4.8}$$

It is well known [3], [9] that $\{C, CP(k)\}_2 = \{S, RP(k) \times RP(k)\}_2 = g([RP(k)]_2)$. Now, k is always even, so $[RP(k)]_2$ is indecomposable in MO_k.

Recall the involution (T', W^2) from above, and set $[V_k]_2 = \varepsilon(\Gamma^{k-2}(\{T', W^2\}_2))$. Now, $[V_k]_2$ is indecomposable in MO_k by 4.2. Thus, $[RP(k)]_2$ and $[V_k]_2$ can differ only by decomposables. We have that $[RP(k)]_2 = [V_k]_2 + [Y_k]_2$, where $[Y_k]_2$ is a sum of decomposable elements. So, we now have that $\{C, CP(k)\}_2 = g([V_k]_2 + [Y_k]_2)$. (4.8) now yields that

$$[P(j,k)]_2 = \sum_{i=0}^{j} [RP(j-i)]_2 \beta(X(i), g([V_k]_2 + [Y_k]_2)).$$

So,

$$\begin{aligned}[P(j,k)]_2 = {} & \beta(X(j), g([V_k]_2)) + \beta(X(j), g([Y_k]_2)) \\ & + \sum_{i=0}^{j-1} [RP(j-i)]_2 \beta(X(i), g([V_k]_2 + [Y_k]_2)).\end{aligned} \tag{4.9}$$

Clearly, $[RP(j-i)]_2 \beta(X(i), g([V_k]_2 + [Y_k]_2))$ is decomposable in MO_* for $0 \leqslant i \leqslant j-1$. By 4.6, $\beta(X(j), g([Y_k]_2))$ is also decomposable. Therefore, since $[P(j,k)]_2$ is indecomposable, $\beta(X(j), g([V_k]_2))$ must also be indecomposable.

From our definition of β, we know that $X(j) \times V_k \times V_k$ is the total space of a covering space for $\beta(X(j), g([V_k]_2))$. Since both $X(j)$ and V_k are closed aspherical manifolds, $\beta(X(j), g([V_k]_2))$ is also a closed aspherical manifold. Thus, $[P(j,k)]_2$ in MO_n, n odd, is equivalent to a closed connected aspherical indecomposable element modulo decomposable elements. Let D_* denote the graded ideal of decomposable elements in MO_*. We have proven the following.

4.10. THEOREM. *For each odd $n \in \mathbf{Z}^+$ there is a closed connected aspherical manifold $M(n)$ such that $[M(n)]_2$ is indecomposable and* $[M(n)]_2 = [P(j,k)]_2$ mod D_*.

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