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CANONICAL OBJECTS IN KIRILLOV THEORY ON NILPOTENT LIE GROUPS

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ABSTRACT. It is shown that to each element f in the dual space of the Lie algebra of a nilpotent Lie group there is a uniquely defined subgroup K_∞ for which the representation corresponding to f is inducible from a square-integrable-modulo-its-kernel representation of K_∞.

I. **Introduction.** Let G be a connected, simply connected nilpotent Lie group with Lie algebra $\mathfrak{A}$. Let U be an irreducible unitary representation of G. Under the Kirillov correspondence U corresponds to a unique orbit $\mathcal{O}$ of the co-adjoint representation ad* of G in the dual space $\mathfrak{A}^*$. The correspondence is obtained by selecting an f in $\mathcal{O}$ and a polarization for f and then forming an appropriate induced representation (see [1]). The polarizations corresponding to a given f are highly nonunique and noncanonical. In this paper we ask the following question: To what extent is it possible to describe U by means of objects *canonically* defined by $\mathcal{O}$ (or f)? Our main result is the following theorem.

THEOREM. *Associated with each $f \in \mathcal{O}$ there is a canonical subalgebra $\mathcal{K}_\infty$ with the following properties*:

(a) *If K_∞ is the corresponding subgroup to $\mathcal{K}_\infty$ and U^∞ is the representation of K_∞ corresponding to $f|\mathcal{K}_\infty$ then* $\operatorname{ind}(K_\infty, G, U^\infty)$ (*the representation induced by* U^∞) *is irreducible and equivalent to U.*

(b) *K_∞ is invariant under any automorphism which fixes f.*

(c) *U_∞ is square integrable modulo its kernel* (*see* [4]).

This theorem seems useful from several points of view. $\mathcal{K}_\infty$ of course must contain polarizations for f. Thus there is a distinguished class of polarizations. Furthermore, let $\mathcal{K}_\infty$ be the radical of $f|\mathcal{K}_\infty$ i.e.

$$\mathcal{K}_\infty = \{X \in \mathcal{K}_\infty | f([X, \mathcal{K}_\infty]) = \{0\}\}.$$

Then $f([\mathcal{K}_\infty \mathcal{K}_\infty]) = 0$. Let H_∞ be the corresponding subgroup. Then $\exp if \circ \log|H_\infty = \chi_\infty$ defines a character of H_∞. Since U^∞ is square integrable modulo its kernel, $\operatorname{ind}(H_\infty, K_\infty, \chi_\infty)$ is primary and quasi-equivalent to U^∞ (see [4]). Hence $\operatorname{ind}(H_\infty, G, \chi_\infty)$ is primary and quasi-equivalent to U. The subgroups H_∞ can be used to describe in a canonical fashion the primary

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projections onto the primary subspaces of nilmanifolds in much the same way that polarizations are used in the character formulas of [2]. We shall go into this in a later paper.

II. **Proofs.** In this section we shall define and study K_∞. K_∞ is defined by means of $\mathcal{K}_\infty$. We shall at first assume only that G is a connected, simply connected, *solvable* Lie group. This will complicate some of our proofs but we feel that it sheds more light on the subject.

Let $f \in \mathfrak{A}^*$ and let $\mathcal{O}(f)$ be its orbit under ad^*. Let

$$\mathcal{S}(f) = \{\lambda \in \mathfrak{A}^* | \mathcal{O}(f) + t\lambda = \mathcal{O}(f) \text{ for all } t \in \mathbf{R}\}.$$

$\mathcal{S}(f)$ is a subspace of $\mathfrak{A}^*$ which is invariant under ad^*. Let $\mathcal{K}(f) = \bigcap \ker\lambda\ (\lambda \in \mathcal{S}(f))$.

LEMMA 1. *$\mathcal{K}(f)$ is an ideal in $\mathfrak{A}$.*

PROOF. $\mathcal{K}(f)$ is invariant under $\mathrm{ad}(\exp tX)$ for all $X \in \mathfrak{A}$. Differentiating we see $[X, \mathcal{K}(f)] \subset \mathcal{K}(f)$. Q.E.D.

Now we define a sequence of subalgebras of $\mathfrak{A}$ as follows:

$$\mathcal{K}_1(f) = \mathcal{K}(f), \qquad \mathcal{K}_k(f) = \mathcal{K}(f|\mathcal{K}_{k-1}(f)).$$

Let $\mathcal{K}_\infty(f) = \bigcap \mathcal{K}_k(f)\ (k \in \mathbf{N})$.

LEMMA 2. *Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be Lie algebras. Let $A: \mathfrak{A}_1 \to \mathfrak{A}_2$ be an automorphism and let $f_2 \in \mathfrak{A}_2^*$ and $f_1 \in \mathfrak{A}_1^*$ be such that $f_2 \circ A = f_1$. Then $\mathcal{K}_\infty(f_2) = A(\mathcal{K}_\infty(f_1))$.*

PROOF. Clearly $A^*(\mathcal{S}(f_2)) = \mathcal{S}(f_1)$. Hence $A(\mathcal{K}_1(f_1)) = \mathcal{K}_1(f_2)$. The result follows by induction since $\mathcal{K}_\infty(f) = \mathcal{K}_\infty(f|\mathcal{K}_1)$. Q.E.D.

We shall require a criterion for deciding when $\mathcal{K}_\infty = \mathfrak{A}$. First we need some notation. If $f \in \mathfrak{A}^*$ and $\mathfrak{M}$ is subspace of $\mathfrak{A}$, let

$$\mathfrak{M}^f = \{X \in \mathfrak{A} | f([X, M]) = 0 \text{ for all } M \in \mathfrak{M}\}.$$

Let $\mathfrak{R} = \mathfrak{A}^f$. Let $R = \{x \in G | \mathrm{ad}\, xf = f\}$.

THEOREM 1. *$\mathcal{K}_\infty \neq \mathfrak{A}$ iff there is a proper ideal $\mathfrak{J}$ of $\mathfrak{A}$ containing $\mathfrak{R}$.*

PROOF. $\mathcal{K}_\infty \neq \mathfrak{A}$ iff $\mathcal{K}_1 \neq \mathfrak{A}$, so it suffices to consider $\mathcal{K}_1$. We claim that $\mathcal{K}_1$ contains $\mathfrak{R}$. Let

$$K = \{x \in G | \mathrm{ad}^* x(f)|\mathcal{K}_1 = f|\mathcal{K}_1\}.$$

Since $\mathcal{K}_1^\perp$ (the annihilator of $\mathcal{K}_1$) is $\mathcal{S}$, we have $f + \mathcal{K}_1^\perp \subset \mathcal{O}(f)$. It follows that $\mathrm{ad}^* K(f) = f + \mathcal{K}_1^\perp$. Hence $\dim R\backslash K = \dim \mathcal{K}_1^\perp$. The Lie algebra $\mathcal{K}$ of K is $\mathcal{K}_1^f$. Then $\dim \mathcal{K} - \dim \mathfrak{R} = \dim R\backslash K = \dim \mathcal{K}^\perp$. The kernel of the map $X \to f([X, \cdot])$ of $\mathcal{K}$ into $\mathfrak{A}^*$ is $\mathfrak{R}$ and hence the image of $\mathcal{K}$ has $\dim \mathcal{K} - \dim \mathfrak{R} = \dim \mathcal{K}_1^\perp$. The image is contained in $\mathcal{K}_1^\perp$ so

$$\mathcal{K}_1^\perp = \{f([X, \cdot])|X \in \mathcal{K}\}.$$

Hence $\mathcal{K}_1 = \mathcal{K}^f$ and thus $\mathfrak{R} \subset \mathcal{K}_1$, proving that $\mathfrak{R}$ is contained in an ideal.

Conversely, let R_0 be the subgroup corresponding to $\mathfrak{R}$. If $\mathcal{I}$ is an ideal containing $\mathfrak{R}$, we claim that $\mathcal{I}^\perp \subset \mathcal{S}(f)$. Let $\mathfrak{L} = \mathcal{I}^f$. Let L be the corresponding analytic subgroup of G. The mapping ϕ: $x \to \mathrm{ad}^* x(f)$ of L into $\mathcal{O}(f)$ maps into the affine subspace $f + \mathcal{I}^\perp$ and is constant on cosets of R_0 in L. (Note that $R_0 \subset L$.) It gives rise to a C^∞ map $\tilde{\phi}$ of $R_0\backslash L$ into $f + \mathcal{I}^\perp$. The tangent space at R_0e in $R_0\backslash L$ is canonically isomorphic with $\mathfrak{R}\backslash\mathfrak{L}$ while the tangent space at f in $f + \mathcal{I}^\perp$ is $\mathcal{I}^\perp$. The differential of $\tilde{\phi}$ at R_0e is given by $X + \mathfrak{R} \to f([X, \cdot])$. Now the bilinear form $B_f = f([\cdot, \cdot])$ is nondegenerate on $\mathfrak{R}\backslash\mathfrak{A}$ and hence $\dim \mathfrak{R}\backslash\mathfrak{A} - \dim \mathfrak{R}\backslash\mathcal{I} = \dim \mathfrak{R}\backslash\mathfrak{L}$ since $\mathfrak{L} = \mathcal{I}^f$. It follows that $\dim \mathfrak{R}\backslash\mathfrak{L} = \dim \mathcal{I}^\perp$ so $d\tilde{\phi}$ is surjective at R_0e. Hence $d\tilde{\phi}$ is nonsingular at R_0e and $\tilde{\phi}$ is an open map in a neighborhood of R_0e. In particular $\mathrm{ad}^* L(f)$ contains a neighborhood of f in $f + \mathcal{I}^\perp$. Similar comments hold for f replaced by $\mathrm{ad}^* g(f)$, $g \in L$. It follows that $\mathrm{ad}^* L(f)$ is open in $f + \mathcal{I}^\perp$. The same is true for any $f' \in f + \mathcal{I}^\perp$ so the orbits of L in $f + \mathcal{I}^\perp$ are all open. Since different orbits are disjoint and $f + \mathcal{I}^\perp$ is a union of orbits, this contradicts the connectedness of $f + \mathcal{I}^\perp$ unless there is only one orbit. Hence $\mathrm{ad}^* L(f) = f + \mathcal{I}^\perp$, showing that $f + \mathcal{I}^\perp \subset \mathcal{O}(f)$. Similarly $f' + \mathcal{I}^\perp \subset \mathcal{O}(f)$ for any $f' \in \mathcal{O}(f)$. It follows that $\mathcal{I}^\perp \subset \mathcal{S}(f)$. Q.E.D.

Recall that a subalgebra $\mathcal{K}$ of $\mathfrak{A}$ is said to be subordinate to f if $f([\mathcal{K}, \mathcal{K}]) = 0$.

COROLLARY. *If $\mathfrak{A}$ is nilpotent, $\mathcal{K}_\infty$ is subordinate to f.*

PROOF. Let $f' = f|\mathcal{K}_\infty$. By definition of $\mathcal{K}_\infty$, $\mathcal{K}_\infty(f') = \mathcal{K}_\infty$. Hence there is no proper ideal $\mathcal{I}$ containing $\mathfrak{R}(f')$. Since $\mathcal{K}_\infty$ is nilpotent, this implies that $\mathfrak{R}(f') = \mathcal{K}_\infty$. Obviously $\mathfrak{R}(f')$ is subordinate. Q.E.D.

DEFINITION. Let $\mathcal{K}_\infty = (\mathcal{K}_\infty)^f \cap$ normalizer $(\mathcal{K}_\infty)$. Let K_∞ be the corresponding connected analytic subgroup.

PROPOSITION. *$\mathcal{K}_\infty = (\mathcal{K}_\infty)^f$ and $\mathrm{ad}^* K_\infty(f) = f + \mathcal{K}_\infty^\perp$.*

PROOF. Let $\mathcal{K}_1 = \mathcal{K}_1^f$ and let K_1 be the corresponding subgroup. For $k \in K_1$, $\mathrm{ad}^* (k)f|\mathcal{K}_1 = f|\mathcal{K}_1$. Hence $\mathrm{ad}^* k(\mathcal{K}_\infty) = \mathcal{K}_\infty$ by Lemma 2. (Recall $\mathcal{K}_\infty = \mathcal{K}_\infty(f|\mathcal{K}_1)$.) It follows that $\mathcal{K}_1 \subset \mathcal{K}_\infty$. Let $\mathcal{K}_n(f) = \mathcal{K}_1(f|\mathcal{K}_{n-1})$. (Note that $\mathcal{K}_n(f) \subset \mathcal{K}_{n-1}(f)$.) Then $\sum_i \mathcal{K}_i \subset \mathcal{K}_\infty$ and $\mathcal{K}_\infty^f \subset \bigcap_i \mathcal{K}_i^f$. As was shown in the proof of the above theorem, $\mathcal{K}_1^f = \mathcal{K}_1$. Hence $\mathcal{K}_i^f \cap \mathcal{K}_{i-1} = \mathcal{K}_i$. It follows that $\bigcap_i \mathcal{K}_i = \bigcap_i \mathcal{K}_i = \mathcal{K}_\infty$. Hence $\mathcal{K}_\infty^f \subset \mathcal{K}_\infty$. Obviously $\mathcal{K}_\infty \subset \mathcal{K}_\infty^f$ so we have shown $\mathcal{K}_\infty^f = \mathcal{K}_\infty$. That $\mathrm{ad}^* K_\infty(f) = f + \mathcal{K}_\infty^\perp$ is the same as the argument that $\mathrm{ad}^* K_1 f = f + \mathcal{K}_1^\perp$ done in the above theorem. Q.E.D.

COROLLARY. *In the notation of the above proof $\mathcal{K}_\infty = \sum_i \mathcal{K}_i$.*

PROOF. Both $\mathcal{K}_\infty$ and $\sum_i \mathcal{K}_i$ contain $\mathfrak{R}$ and both have the same orthogonal complement under B_f. Hence they have the same dimension. Q.E.D.

COROLLARY. *Let* $f_\infty = f|\mathcal{K}_\infty$. *Then the* K_∞ *orbit of* f_∞ *is an affine subspace of* $\mathcal{K}_\infty^*$.

Now we once again assume that G is nilpotent. Let $f_\infty = f|\mathcal{K}_\infty$ and let U^∞ be the irreducible representation of K_∞ corresponding to f_∞ under the Kirillov correspondence.

THEOREM 2. $\operatorname{ind}(K_\infty, G, U^\infty)$ *is irreducible and corresponds to* $\mathcal{O}(f)$ *under the Kirillov correspondence. Also* U^∞ *is square integrable modulo its kernel.*

PROOF. From the above corollary f_∞ has a flat orbit so it follows from the Moore-Wolf theorem [4] that U^∞ is square integrable modulo its kernel.

To prove the irreducibility let $\mathcal{K}_1(f)$ and $\mathcal{K}_1(f)$ be as before. Let H_1 and K_1 be the corresponding subgroups. Let $K_{\infty,1} = K_\infty(f|\mathcal{K}_1)$. K_1 fixes $f|\mathcal{K}_1$ so K_1 normalizes $\mathcal{K}_\infty$ and hence K_1 normalizes $K_{\infty,1}$. From Corollary 3 it follows that $K_\infty = K_1 K_{\infty,1}$. Let $U^{\infty,1}$ be the representation of $K_{\infty,1}$ corresponding to $f|\mathcal{K}_{\infty,1}$ and let $U^1 = \operatorname{ind}(K_{\infty,1}, H_1, U^{\infty,1})$. By induction U^1 is irreducible and corresponds to $f_1 = f|\mathcal{K}_1$. Let L be the stabilizer of U^1 in G –i.e.

$$L = \{x \in G | U^1(x \cdot x^{-1}) \approx U^1\}.$$

$x \in L$ if $\operatorname{ad}^* x(f_1)$ is in the H_1 orbit of f_1 –i.e. iff $\operatorname{ad}^* x(f_1) = \operatorname{ad}^* y(f_1)$ for some $y \in H_1$. This is equivalent to saying $\operatorname{ad}^* y^{-1}x(f_1) = f_1$ –i.e. $x \in K_1' H_1$ where $K_1' = \{x \in G | \operatorname{ad} x(f_1) = f_1\}$. It follows from standard arguments that K_1' is connected (see [1, I.3.3], e.g.) and that the Lie algebra of K_1' is precisely $\mathcal{K}_1^f = \mathcal{K}_1$. Hence $K_1' = K_1$ and $L = K_1 H_1$. U^1 extends to a representation V of L. In fact, $K_\infty = K_1 K_{\infty,1} \subset L$. Let $V = \operatorname{ind}(K_\infty, L, U^\infty)$. Since $L = K_1 H_1$, $V|H_1 = \operatorname{ind}(K_{\infty,1}, H_1, U^{\infty,1}) = U^1$. It now follows from Mackey's theorem [3] that $\operatorname{ind}(L, G, V)$ is irreducible. From the theorem on inducing in stage this is just $\operatorname{ind}(K_\infty, G, U^\infty)$. It is obvious that this representation corresponds to f –simply pick a polarization for f contained in $\mathcal{K}_\infty$. Q.E.D.

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