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CLUSTER SETS ON OPEN RIEMANN SURFACES

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ABSTRACT. Generalizations of theorems of Gross-Iversen type on exceptional values are given for analytic mappings on open Riemann surfaces.

The classical, well-known theorems of Picard and Iversen (cf. [2, p. 3]) concerning isolated, essential singularities were extended to those of Hällström and Cartwright (cf. [2, pp. 10, 15]) concerning essential singularities of capacity zero, respectively. These extensions were furthermore extended to the theorems of Tsuji and Noshiro (cf. [2, pp. 14, 19]) which are stated as follows:

Let D be an arbitrary domain, B its boundary, A a compact set of capacity zero on B and z_0 a point of A. Let $\varphi(z)$ be single-valued and meromorphic in D. $C_D(\varphi, z_0)$ and $C_{B-A}(\varphi, z_0)$ denote the full clsuter set of $\varphi(z)$ at z_0 and the boundary cluster set of $\varphi(z)$ at z_0 modulo A, respectively.

(1) Every value of $C_D(\varphi, z_0) - C_{B-A}(\varphi, z_0)$ is assumed by $\varphi(z)$ infinitely often in any neighborhood of z_0 except for a possible set of values of capacity zero.

(2) If $\alpha \in C_D(\varphi, z_0) - C_{B-A}(\varphi, z_0)$ is an exceptional value of $\varphi(z)$ in a neighborhood of z_0, then either α is an asymptotic value of $\varphi(z)$ at z_0 or there is a sequence $\xi_n \in A$ $(n = 1, 2, 3, \dots)$ converging to z_0 such that α is an asymptotic value of $\varphi(z)$ at each ξ_n.

In this paper, (1) and (2) will be generalized for analytic mappings from open Riemann surfaces into Riemann surfaces. These generalizations will be given by Theorem 1 and Theorem 2 .

Let f be an analytic mapping from an open Riemann surface R into a Riemann surface S. Let R^* and S^* denote a metrizable compactification and an arbitrary compactification of R and S, respectively. $\overline{X}$ and bdy X mean the closure and the boundary of a subset X of R^* or S^* with respect to R^* or S^*, respectively. ∂X means the relative boundary of a subset X of R or S with respect to R or S.

We write $\Delta = R^* - R$. The full cluster set of f at $p \in \Delta$ is defined as $C(f, p) = \bigcap_{r>0} \overline{f(U(p, r) \cap R)}$, where $U(p, r)$ denotes the r-neighborhood of p. For a set E on Δ, the boundary cluster set of f at p modulo E is defined

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as $C_{\Delta-E}(f,p) = \bigcap_{r>0}\overline{\bigcup_{q\in W(p,r)}C(f,q)}$, where $W(p,r) = U(p,r) \cap \Delta - E - \{p\}$. It is said that a path $\gamma(t)$ $(0 \leqslant t < 1)$ in R tends to a connected set K $(\subset \Delta)$, when for any r-neighborhood $U(K,r)$ of K, there is a $t(K,r)$ such that $\gamma(t) \subset U(K,r)$ for all $t \geqslant t(K,r)$. Henceforth let $V(P)$, $V_0(P)$ and $V^*(P)$ denote parametric disks about a point P of R or S.

THEOREM 1. *Let E be a polar set on Δ and p a point of E. Let $r_1 > r_2 > \cdots > r_n > \ldots, r_n \to 0$. If $E \cap$ bdy $U(p,r_n) = \varnothing$ for every n, then every point of $F = C(f,p) \cap S - C_{\Delta-E}(f,p)$ is assumed by f infinitely often in any $U \in \{U(p,r)\}$, with a possible exceptional set of capacity zero.*

PROOF. Let $f_{U\cap R}$ denote the restriction of f to $U \cap R$ and $n(f_{U\cap R}, b)$ the number of the points of $f_{U\cap R}^{-1}(b)$ for each $b \in S$, where each point is counted with its multiplicity. $n(f_{U\cap R}, b)$ is lower semicontinuous on S and, hence, $F_n = \{b \in F;\ n(f_{U\cap R}, b) \leqslant n\}$ $(n = 0, 1, 2, \ldots)$ is relatively closed in the open set $S - C_{\Delta-E}(f,p)$. Suppose that $\{b \in F;\ n(f_{U\cap R}, b) < \infty\}$ is of positive capacity. Then there is an N $(0 \leqslant N < \infty)$ for which F_{N-1} is of capacity zero and F_N is of positive capacity, where $F_{-1} = \varnothing$. It is possible to find some $c \in F_N - F_{N-1}$, which is not a branch point, such that for any $V(c)$, $V(c) \cap F_N$ is of positive capacity.

First consider the case where $U(p,r) \cap (\Delta - E) \neq \varnothing$ for every $U(p,r)$. Then $C_{\Delta-E}(f,p) \neq \varnothing$. Choose an open set G $(\not\ni c)$ containing $C_{\Delta-E}(f,p)$. There are $V(a_i)$ $(\subset U)$ $(i = 1, 2, \ldots, N)$ such that $f(a_i) = c$ and $V(a_i) \cap V(a_j) = \varnothing$ $(i \neq j)$, and which are mapped onto a $V_0(c)$, satisfying $V_0(c) \cap G = \varnothing$, by f homeomorphically. There is a $U' \in \{U(p,r)\}$ such that $U' \subset U$ and $\overline{\bigcup_{q\in W}C(f,q)} \subset G$, where $W = U' \cap \Delta - E$. For each $q \in W$, there is a $U(q) \in \{U(q,r)\}$ satisfying $f(U(q) \cap R) \subset G$. Since $\overline{f(\bigcup_{q\in W}(U(q) \cap R))} \cap V_0(c) = \varnothing$, $(\bigcup_{q\in W}(U(q) \cap R)) \cap f^{-1}(V_0(c)) = \varnothing$ and, hence, $f^{-1}(V_0(c)) \cap W = \varnothing$. Therefore $f^{-1}(V_0(c)) \cap U' \cap \Delta \subset E$.

Choose a $U(p, r_{N^*})$ such that $\overline{U(p,r_{N^*})} \subset U'$ and $\overline{U(p,r_{N^*})} \cap \overline{V(a_i)} = \varnothing$ $(i = 1, 2, \ldots, N)$. $\overline{f(U(p,r_{N^*}) \cap R)}$ contains c, because $c \in C(f,p)$. Since $E \cap$ bdy $U(p,r_{N^*}) = \varnothing$, it is easy to see that $f^{-1}(\overline{V_0(c)}) \cap$ bdy $U(p,r_{N^*})$ is compact in R. Hence $f(R \cap$ bdy $U(p,r_{N^*}))$ $(\not\ni c)$ is relatively closed on $\overline{V_0(c)}$. Thus it is possible to choose some $V^*(c)$ $(\subset V_0(c))$ satisfying $\overline{V^*(c)} \cap f(R \cap$ bdy $U(p,r_{N^*})) = \varnothing$.

Take a component D^* $(\subset U(p,r_{N^*}))$ of $f^{-1}(V^*(c))$. $\partial f(D^*) - \partial V^*(c)$ has regular points relative to the Dirichlet problem (cf. [1, pp. 42, 50]). Let h be a continuous function with the property that h is equal to 0 on $\partial f(D^*) \cap \partial V^*(c)$ and $0 < h \leqslant 1$ in $\partial f(D^*) - \partial V^*(c)$. Let u be the solution of the Dirichlet problem in $f(D^*)$ with h as its boundary function. Then $0 < u < 1$ in $f(D^*)$ and $\lim_{D^*\ni a\to q} u \circ f_{D^*}(a) = 0$ at every $q \in \partial D^*$. Since E is polar, there is a positive superharmonic function s on R with $\lim_{a\to q} s(a) = \infty$ at every $q \in E$. Therefore, for any $\varepsilon > 0$, $\underline{\lim}_{D^*\ni a\to q}(\varepsilon s_{D^*}(a) - u \circ f_{D^*}(a)) \geqslant 0$ at every $q \in (\partial D^*) \cup (\overline{D^*} \cap \Delta)$. It follows from the minimum principle (cf. [1, p. 11]) that $-u \circ f_{D^*} \geqslant 0$ in D^*. This implies a contradiction.

Next consider the case where $U(p, r) \cap (\Delta - E) = \emptyset$ for some $U(p, r)$. Then $C_{\Delta - E} = \emptyset$. We have a contradiction, as we see easily from the above proof. Thus the proof of Theorem 1 is complete.

THEOREM 2. *If, under the hypotheses of Theorem* 1, $e \in F$ *is an exceptional point of* f *in some* $U^* \in \{U(p, r)\}$, *then either* e *is an asymptotic point of* f *at* p *or there is an infinite sequence of connected sets* K_n $(\subset E)$ *converging to* p *such that* e *is the asymptotic point of* f *along a path tending to* K_n.

PROOF. First consider the case where $U(p, r) \cap (\Delta - E) \neq \emptyset$ for every $U(p, r)$. Take any $U'' \in \{U(p, r)\}$ contained in U^*. Since $n(f_{U^* \cap R}, e) = 0$, it is possible to take U', $U(p, r_{N^*})$ and $V^*(e)$ in the proof of Theorem 1 such that $\overline{U(p, r_{N^*})} \subset U' \cap U''$ and $\overline{V^*(e)} \cap f(R \cap \text{bdy } U(p, r_{N^*})) = \emptyset$. Any component D^* $(\subset U(p, r_{N^*}))$ of $f^{-1}(V^*(e))$ is not relatively compact in R.

Let $g_{V^*(e)}(b, e)$ denote the Green's function for $V^*(e)$ with pole at e. Suppose that $\overline{f(D^*)} \not\ni e$. Then $g_{V^*(e)}(f_{D^*}(a), e)$ is bounded on D^*. As in the proof of Theorem 1, it follows that $-g_{V^*(e)}(f_{D^*}(a), e) \geqslant 0$ in D^*. This is impossible. Therefore $\overline{f(D^*)} \not\ni e$.

Let $w = \psi(b)$ be a local parameter of $V^*(e)$, and write $\psi(V^*(e)) = \{w;\ |w| < 1\}$ $(\psi(e) = 0)$ and $W_{1/n} = \{w;\ |w| < 1/n\}$ $(n = 1, 2, 3, \ldots)$. Let $\{D_n\}$ be an infinite sequence of components of $f^{-1} \circ \psi^{-1}(W_{1/n})$ such that $D_{n+1} \subset D_n \subset D^*$, and $\{p_n\}$ an infinite sequence of points $p_n \in D_n$. D_n is not relatively compact in R and $\overline{f(D_n)} \ni e$. For any compact set K' $(\subset R)$, there is an N_0 such that $D_n \subset R - K'$ for all $n \geqslant N_0$. Furthermore, as in the proof of Theorem 1, $\overline{D}_n \subset E$. For each n, there is a simple arc λ_n joining p_n to p_{n+1} in D_n such that $f(\lambda_n) \subset \psi^{-1}(W_{1/n})$. Thus the path $\lambda = \bigcup \lambda_n$ tends to a component of $E \cap U(p, r_{N^*})$ and has the property that $f \to e$ along λ. Since $E \cap \text{bdy } U(p, r_n) = \emptyset$ for every n, our assertion is proved.

Next consider the case where $U(p, r) \cap (\Delta - E) = \emptyset$ for some $U(p, r)$. From the above proof, we see easily that our conclusion holds. Thus the proof of Theorem 2 is complete.

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