PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 66, Number 1, September 1977

CONDITIONS FOR GENERATING A NONVANISHING BOUNDED ANALYTIC FUNCTION

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ABSTRACT. B. A. Taylor and L. A. Rubel have posed the problem of finding necessary and sufficient conditions on a set of given functions $f_1, f_2, \ldots, f_n$ in H^∞ such that there exist functions $g_1, g_2, \ldots, g_n$ in H^∞ with $\sum_{i=1}^n f_i g_i \neq 0$ in the open unit disc. L. A. Rubel has conjectured that a necessary and sufficient condition is that there exist a harmonic minorant of $\log[\sum_{i=1}^n |f_i|]$ in the open unit disc. The major result of this paper proves that the conjecture is true if one of the given functions $f_1, f_2, \ldots, f_n$ has a zero set which is an interpolation set for H^∞.

Let D be the open unit disc in the complex plane and let H^∞ denote the space of bounded holomorphic functions on D. If $f_1, f_2, \ldots, f_n$ are n given functions in H^∞, we seek necessary and sufficient conditions that there are functions $g_1, g_2, \ldots, g_n$ in H^∞ with

$$\sum f_i(z) g_i(z) \neq 0, \qquad z \in D.\text{[1]} \tag{1}$$

If it happens that $|\sum f_i g_i|$ is actually bounded away from zero on D, then $f_1, f_2, \ldots, f_n$ generate H^∞ (as an ideal); it is a known and difficult theorem of L. Carleson that $f_1, f_2, \ldots, f_n$ generate H^∞ if and only if

$$\sum |f_i(z)| \geqslant \delta > 0, \qquad z \in D \tag{2}$$

(see [2, p. 163]). Since we ask here for only the weaker condition that $\sum f_i g_i$ does not vanish in D, it is to be expected that (2) will be replaced by some weaker condition. L. A. Rubel has conjectured that (1) holds for some $g_1, g_2, \ldots, g_n$ if and only if the function

$$\mu(z) = \log \sum |f_i(z)| \tag{3}$$

has a harmonic minorant on D.[2] We show in what follows that this conjecture is true under the additional hypothesis that the zero set of some f_j is an interpolation sequence for H^∞.

A sequence of points $\{z_k\}_{k=1}^\infty$ in D is called an interpolation sequence for H^∞ if, for each bounded sequence of complex numbers $\{w_k\}_{k=1}^\infty$, there exists a function f in H^∞ such that $f(z_k) = w_k$ for every k. A Blaschke product is an

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Received by the editors March 10, 1977.
AMS (MOS) subject classifications (1970). Primary 30A76, 30A78, 30A82; Secondary 30A08.
Key words and phrases. Blaschke product, H^∞, interpolation sequence for H^∞, harmonic minorant, zero set, generating a nonvanishing bounded analytic function.

[1]All summations will be indexed from $i = 1$ to $i = n$ unless otherwise indicated.

[2]Private communications.

analytic function B of the form

$$B(z) = z^p \prod_{k=1}^{\infty} \left[\frac{\bar{\alpha}_k \cdot (\alpha_k - z)}{|\alpha_k| \cdot (1 - \bar{\alpha}_k z)} \right]^{p_k}$$

where

(i) $p, p_1, p_2, \ldots$ are nonnegative integers;

(ii) the α_k are distinct nonzero numbers in D;

(iii) the product $\prod_{k=1}^{\infty} |\alpha_k|^{p_k}$ is convergent.

For further discussion on interpolation sequences and Blaschke products, see [**2**, pp. 66–74, 194–207] and [**1**, pp. 18–29, 136–143].

Suppose $\sum f_i g_i \neq 0$ in D where the f_i's and g_i's are in H^∞. Then $\sum f_i g_i = e^h$ where h is analytic in D.

Consequently, $e^{\operatorname{Re} h} \leqslant c\sum|f_i|$ and so $\operatorname{Re} h - \log c \leqslant \log \sum|f_i|$ and $\operatorname{Re} h - \log c$ is harmonic. This shows that the condition that $\log \sum|f_i|$ have a harmonic minorant is necessary. It is clearly sufficient for the case $n = 1$. For $n \geqslant 2$, suppose the zero set of some f_j is finite. If $\mu(z)$ has a harmonic minorant, then the zero sets of $f_1, f_2, \ldots, f_n$ are mutually disjoint. Since the only possible limit points of the zero sets lie on the unit circle, there exists an open set V containing the zero set of f_j such that $|f_j| \geqslant \delta_1 > 0$ outside V and $\sum_{k \neq j} |f_k| \geqslant \delta_2 > 0$ on $\bar{V}$. Consequently, $\sum|f_i| \geqslant \delta > 0$ on the open disc, and by the result of L. Carleson cited above, $f_1, f_2, \ldots, f_n$ generate H^∞. Hence we now restrict our attention to the case where $n \geqslant 2$ and the zero sets of the f_i's are all infinite.

THEOREM. *Given $f_1, f_2, \ldots, f_n$ in H^∞ with the zero set of f_1 an interpolation sequence for H^∞, there exist functions $g_1, g_2, \ldots, g_n$ in H^∞ with $\sum f_i g_i \neq 0$ in D if and only if* $\log \sum|f_i|$ *has a harmonic minorant.*

We need a preliminary result:

LEMMA. *Let B be a Blaschke product whose zero set $\{z_k\}_{k=1}^{\infty}$ is an interpolation sequence for H^∞. Let $f_1, f_2, \ldots, f_n$ be functions in H^∞. Then $I_1 = I_2$ where*

$$I_1 = \left\{ Bg_0 + \sum f_i g_i \colon g_0, g_1, g_2, \ldots, g_n \text{ are in } H^\infty \right\}$$

and

$$I_2 = \left\{ g \in H^\infty \colon |g(z_k)| \Big(\sum |f_i(z_k)| \Big)^{-1} \leqslant M \text{ for all } k \text{ where } M \text{ depends on } g \right\}.$$

PROOF OF LEMMA. Let $F = Bg_0 + \sum f_i g_i$ lie in I_1. Then

$$\Big|B(z_k) g_0(z_k) + \sum f_i(z_k) g_i(z_k)\Big| \Big(\sum |f_i(z_k)| \Big)^{-1}$$
$$= \Big|\sum f_i(z_k) g_i(z_k)\Big| \Big(\sum |f_i(z_k)| \Big)^{-1} \leqslant \|g\|_\infty$$

for all k. Hence $I_1 \subset I_2$.

Now suppose $g \in I_2$. For $i = 1, 2, \ldots, n$ and $k = 1, 2, 3, \ldots,$ define α_{ik} by

$$\alpha_{ik} = \begin{cases} 0 & \text{if } f_i(z_k) = 0, \\ \operatorname{sgn} f_i(z_k) & \text{if } f_i(z_k) \neq 0. \end{cases}$$

Then $\{\alpha_{ik}\}_{k=1}^{\infty} \in l^{\infty}$ for $i = 1, 2, \ldots, n$. Since $\{z_k\}_{k=1}^{\infty}$ is interpolating, there exist $g_1, g_2, \ldots, g_n$ in H^{∞} such that $g_i(z_k) = \alpha_{ik}$ for $i = 1, 2, \ldots, n$ and $k = 1, 2, 3, \ldots$. Consequently,

$$|g(z_k)|\left(\sum f_i(z_k) g_i(z_k)\right)^{-1} = |g(z_k)|\left(\sum |f_i(z_k)|\right)^{-1} \leqslant M$$

for all k. So $\{g(z_k)(\sum f_i(z_k) g_i(z_k))^{-1}\}_{k=1}^{\infty} \in l^{\infty}$, and there exists $h \in H^{\infty}$ such that

$$h(z_k) = g(z_k)\left(\sum f_i(z_k) g_i(z_k)\right)^{-1}$$

for every k. We have that $g - h\sum f_i g_i = 0$ on $\{z_k\}_{k=1}^{\infty}$ which implies that $g - h\sum f_i g_i = Bh_1$, where $h_1 \in H^{\infty}$. Thus $g = Bh_1 + \sum f_i g_i h$ and so $g \in I_1$. This shows that $I_2 \subset I_1$, and the proof is complete.

Proof of Theorem. Let B_1 be the Blaschke factor of f_1. Suppose $\log \sum |f_i|$ has harmonic minorant μ. Let v be a harmonic conjugate of μ. Then $\sum |f_i| \geqslant e^{\mu} = |e^{\mu + iv}|$ and thus $e^{\mu + iv} \in H^{\infty}$. Let $h = \mu + iv$ and let $\{z_k\}_{k=1}^{\infty}$ be the zero set of B_1. Then $|e^{h(z_k)}|(\sum |f_i(z_k)|)^{-1} \leqslant 1$ for all k. By the Lemma, there exist functions $g_0, g_1, \ldots, g_n$ in H^{∞} such that $Bg_0 + \sum f_i g_i = e^h$. Now $f_1 = B_1 G_1$, where $G_1 \in H^{\infty}$ and $G_1 \neq 0$ in D. Hence $B_1 G_1 g_0 + \sum f_i g_i G_1 = G_1 e^h$, and the proof is complete.

The condition in the Theorem that $\{z_k\}_{k=1}^{\infty}$ be an interpolation sequence can be weakened so that $\{z_k\}_{k=1}^{\infty}$ is the union of a finite number of interpolation sequences. The argument is by induction on the number of interpolation sequences and is straightforward. P. J. McKenna has announced the following related result: Let $S = \{z_n\}$ be a sequence of points in the open unit disc satisfying the Blaschke condition. Let μ be the discrete measure concentrated on the sequence S, with weights $\mu(\{z_n\}) = 1 - |z_n|^2$, $n = 1, 2, 3, \ldots$. Then μ is a Carleson measure if and only if S is a finite union of interpolating sequences.

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