

A NEW PROOF OF CARISTI'S FIXED POINT THEOREM

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ABSTRACT. An elementary proof of Caristi's generalization of the contraction theorem is presented.

1. Introduction. In the proof of the contraction theorem, one is presented with a construction of the fixed point as a countable iteration of applications of the given operator to some other point of the space. This construction allows one to infer properties of the fixed point from those of the initial point of the iteration.

Recently, based on the work of Brøndsted [1] (see also [4]), Caristi [3] has given a significant generalization of the contraction theorem. However, Caristi's proof, as well as others more recent [2], [5], lack the constructive aspect of the original proof.

Below we present a version of Caristi's theorem which offers a construction of the fixed point as a countable iteration of application of suitable operators. The theorem we present is proved by a very simple argument which is in the spirit of the original contraction theorem.

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1.1 FIXED NOTATION. Let (M, d) be a complete metric space. Let $\phi: M \rightarrow R^+$ (nonnegative reals) and $g: M \rightarrow M$ be not necessarily continuous functions such that $d(x, g(x)) \leq \phi(x) - \phi(g(x))$, $x \in M$.

Given a sequence of functions f_i , $1 \leq i < \infty$, set

$$\prod_{i=1}^{\infty} f_i(x) = \lim_{i \rightarrow \infty} f_i f_{i-1} \cdots f_1(x)$$

if it exists, call it the countable composition of the f_i .

1.2 DEFINITION. Let $\Phi = \{f|f: M \rightarrow M \text{ and } d(x, f(x)) \leq \phi(x) - \phi(f(x))\}$. Let $\Phi_g = \{f|f \in \Phi \text{ and } \phi(f) \leq \phi(g)\}$.

1.3 LEMMA. *Both Φ and Φ_g are closed under compositions, and if ϕ is lower semicontinuous, under countable compositions.*

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PROOF. First observe that Φ and Φ_g are closed under compositions. Let f_1 and f_2 be in

$$\begin{aligned} \Phi d(x, f_2 f_1(x)) &\leq d(x, f_1(x)) + d(f_1(x), f_2 f_1(x)) \\ &\leq (\phi(x) - \phi(f_1(x))) + (\phi(f_1(x)) - \phi(f_2 f_1(x))) \\ &= \phi(x) - \phi(f_2 f_1(x)) \end{aligned}$$

which shows $f_2 f_1 \in \Phi$. If $f_1 \in \Phi_g$ since $\phi(f_1(x)) - \phi(f_2 f_1(x)) \geq 0$ we have $\phi(f_2 f_1) \leq \phi(g)$,

To show the classes are closed under countable composition we use the following lemma.

LEMMA. Let $\{x_i\}$ be a sequence in M . $\exists. d(x_i, x_{i+1}) \leq \phi(x_i) - \phi(x_{i+1})$ all i ; then $\lim_{i \rightarrow \infty} x_i = \bar{x}$ exists and $d(x_i, \bar{x}) \leq \phi(x_i) - \phi(\bar{x})$ for each i .

PROOF. Since the sequence of values $\phi(x_i)$ is decreasing and bounded below (by 0) and $d(x_i, x_j) \leq \phi(x_i) - \phi(x_j)$ for $i \leq j$, $\{x_i\}$ is a Cauchy sequence in M . Let $\bar{x} = \lim x_i$. On the other hand

$$d(x_i, \bar{x}) = \lim_{j \rightarrow \infty} d(x_i, x_j) \leq \phi(x_i) - \lim_{j \rightarrow \infty} \phi(x_j) \leq \phi(x_i) - \phi(\bar{x}).$$

This last inequality by lower semicontinuity.

The remainder of the proof of the theorem amounts to the observation that for each $x \in M$ the sequence $x_i = f_i f_{i-1} \cdots f_1(x)$ satisfies the conditions of the lemma.

Before stating the main theorem of this note we also need the following:

1.4 DEFINITION. (1) For $A \subseteq X$ define the *diameter* of A , written

$$D(A) = \text{lub}_{x_i, x_j \in A} (d(x_i, x_j)).$$

(2) Set $r(A) = \text{glb}_{x \in A} (\phi(x))$; note $B \subseteq A$ implies $r(B) \geq r(A)$.

(3) Let $\Phi' \subseteq \Phi$. For each $x \in X$ define $S_x = \{f(x) | f \in \Phi'\}$.

1.5 LEMMA. $D(S_x) \leq 2(\phi(x) - r(S_x))$.

PROOF.

$$\begin{aligned} d(f_1(x), f_2(x)) &\leq d(x, f_1(x)) + d(x, f_2(x)) \\ &\leq \phi(x) - \phi(f_1(x)) + \phi(x) - \phi(f_2(x)) \\ &\leq 2(\phi(x) - r(S_x)). \end{aligned}$$

We are now able to state our version of Caristi's theorem.

1.6 THEOREM. Let $\Phi' \subseteq \Phi$ be closed under compositions. Let $x_0 \in M$.

(a) Let Φ' also be closed under countable compositions. Then there exists an $\bar{f} \in \Phi'$ such that $\bar{x} = \bar{f}(x_0)$ and $g(\bar{x}) = \bar{x}$ for all $g \in \Phi'$.

(b) Let the members of Φ' be continuous functions. Then there exists a sequence of functions $f_i \in \Phi'$ and $\bar{x} = \lim_{i \rightarrow \infty} f_i f_{i-1} \cdots f_1(x_0)$ that $g(\bar{x}) = \bar{x}$ for all $g \in \Phi'$.

PROOF. Let ε_i be a positive sequence converging to 0. Select $f_1 \in \Phi'$. \exists . $\phi(f_1(x_0)) - r(S_{x_0}) < \varepsilon/2$. Set $x_1 = f(x_0)$. Since Φ' is closed under composition we have $S_{x_1} \subseteq S_{x_0}$ and $D(S_{x_1}) \leq 2(\phi(x_1) - r(S_{x_1})) \leq 2(\phi(f_1(x_0)) - r(S_{x_0})) < \varepsilon_1$.

Repeating this process we obtain a sequence of f_i such that $x_{i+1} = f_i(x_i)$, $S_{x_{i+1}} \subseteq S_{x_i}$ and $D(S_{x_i}) < \varepsilon_i$.

Under hypothesis (a), let $\bar{f} = \prod_{i=1}^{\infty} f_i$ and $\bar{x} = \bar{f}(x_0)$. $\bar{x} \in S_{x_i}$ for each i since $\bar{x} = \prod_{j=i+1}^{\infty} f_j(x_i)$. On the other hand, since $\lim_{i \rightarrow \infty} D(S_{x_i}) = 0$ we have $\bar{x} = \bigcap_{i=0}^{\infty} S_{x_i}$.

We must now only check that $g(\bar{x}) = \bar{x}$. But again $g(\bar{x}) \in S_{x_i}$ for each i since $g(\bar{x}) = g(\prod_{j=i+1}^{\infty} f_j(x_i))$.

Under hypothesis (b), let $\bar{x} = \lim_{i \rightarrow \infty} f_i f_{i-1} \cdots f_1(x_0) = \lim_{i \rightarrow \infty} x_i$. Firstly, since $\{x_j\}_{j>i} \subseteq S_i$ for each i we have that $\bar{x} \in \bar{S}_i$, the closure of S_i . Since $D(\bar{S}_i) = D(S_i)$ we have that $\bar{x} = \bigcap_{i=0}^{\infty} \bar{S}_i$.

To verify that $g(\bar{x}) = \bar{x}$, observe that $g(x_i) \in S_{x_i}$ for each i . Hence, for any $\varepsilon > 0$ there exists i_0 such that $B_\varepsilon(g(\bar{x})) \cap S_{x_i} \neq \emptyset$, $i > i_0$ (here we need g to be continuous). Therefore, for $i > i_0$, $d(g(\bar{x}), \bar{x}) < \varepsilon + \varepsilon_i$, and since $\varepsilon_i \rightarrow 0$ we have that $d(g(\bar{x}), \bar{x}) \leq \varepsilon$, ε arbitrary.

1.7. FINAL REMARKS. In 1.6(b) one may choose $\Phi' = \{g^n\}$, the set consisting of g and its finite iterates. For this choice of Φ' one has $\bar{x} = \lim_{n \rightarrow \infty} g^n(x_0)$ as in the contraction theorem.

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