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A UNIQUE CONTINUATION THEOREM INVOLVING A DEGENERATE PARABOLIC OPERATOR

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ABSTRACT. We consider the degenerate parabolic operator $L[u] = \gamma B[u] - u_t$ on a domain $D = \Omega \times (0, T]$ where $B[u] = \sum_{i,j=1}^{n}(a_{ij}(x)u_{x_j})_{x_i}$ and γ is an arbitrary complex number. Classically, $\gamma = 1$ and the real-valued matrix (a_{ij}) is positive definite. We assume (a_{ij}) is a real-valued symmetric matrix but not necessarily definite. We prove that any complex-valued function u which satisfies the inequality $|L[u]| \leqslant c|u|$ for some nonnegative constant c and vanishes initially as well as on the boundary of Ω must vanish on all of D. The theorem is particularly useful in studying uniqueness for many systems which are not parabolic.

1. **Introduction.** We consider the differential operator

$$L \equiv \gamma B - \partial/\partial t, \tag{1.1}$$

where γ is a complex number, and

$$B[u] \equiv \sum_{i,j=1}^{n} \left(a_{ij}(x)u_{x_j}\right)_{x_i}, \tag{1.2}$$

in which the real-valued matrix $(a_{ij}(x))$ defined on a domain Ω is assumed to be symmetric but not necessarily definite. We are using subscripts to denote differentiation.

A unique continuation theorem is established for the differential inequality

$$|L[u]|^2 \leqslant c|u|^2 \tag{1.3}$$

where c is a nonnegative constant. Such results are useful in the study of uniqueness for systems of equations of the form (See [1].)

$$u_t^m = \sum_{l=1}^{N} \sum_{i,j=1}^{n} \left(a_{ij}(x)e^{lm}u_{x_j}^l\right)_{x_i} + f_m(x, t, u) \tag{1.4}$$

for $1 \leqslant m \leqslant N$. In the event that the constant matrix (e^{lm}) has complex eigenvalues, classical results are not applicable to the system (1.4).

The problem of unique continuation for linear parabolic equations was considered by Lees and Protter [4] who generalized the results of several authors by considering differential inequalities of the form

$$(L[v])^2 \leqslant c_1 t^{-2} v^2 + c_2 \sum_{m=1}^{n} (v_{x_m})^2 \tag{1.5}$$

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in which it was assumed that $\gamma = 1$ in (1.1) and that the matrix (a_{ij}) be time dependent and satisfy the usual parabolicity condition of being positive definite.

We consider complex-valued functions $u(x_1, \ldots, x_n, t)$ which satisfy (1.3) and which vanish initially as well as on the boundary of Ω. The proofs require considerably sharper integral estimates than those used by Lees and Protter [4] although the methods of the proofs are essentially the same.

2. **The main results.** Let $D = \Omega \times (0, T]$ where Ω is a bounded open connected set in real n-dimensional Euclidean space, R^n, with a sufficiently smooth boundary $(\partial \Omega)$ to ensure the validity of integration by parts.

We shall say that the differential operator L defined in (1.1) and (1.2) satisfies condition (G) if the real-valued symmetric matrix (a_{ij}) is continuous on $\bar{\Omega}$ with continuous bounded first order derivatives on Ω. The usual parabolicity condition that the matrix (a_{ij}) be (positive) definite is not required here. We shall use the notation

$$(v, w) = \int_D v(x, t)\bar{w}(x, t)dx\, dt, \tag{2.1}$$

$$\|v\| = (v, v)^{1/2}. \tag{2.2}$$

We let $\tilde{P}$ be the set of complex-valued functions v such that v is continuous on $\bar{D}$, the second partial derivatives of v in x are continuous and square integrable on D, and the first partial derivatives of v_t in x are continuous and square integrable on D. We let P be the set of funtions in $\tilde{P}$ which vanish on the boundary of D. Finally we let $\lambda(t) = t + \eta$ where η is any fixed positive number.

Before proving the main result of the paper, we need two preliminary lemmas.

LEMMA 1. *If the operator L satisfies condition* (G), $u \in P$, *k is any positive integer and* $z(x, t) = [\lambda(t)]^{-k}u(x, t)$, *then*

$$\|\gamma B[z]\|^2 + \|z_t\|^2 + k^2\|\lambda^{-1}z\|^2 \geqslant 2\mathrm{Re}(\gamma B[z], z_t + k\lambda^{-1}z). \tag{2.3}$$

PROOF. We let $v = \mathrm{Re}(z)$, $w = \mathrm{Im}(z)$, $\alpha = \mathrm{Re}(\gamma)$ and $\beta = \mathrm{Im}(\gamma)$. We now obtain

$$\begin{aligned}\mathrm{Re}(\gamma B[z], z_t) &= \alpha(B[v], v_t) + \alpha(B[w], w_t)\\ &\quad + \beta(B[v], w_t) - \beta(B[w], v_t).\end{aligned} \tag{2.4}$$

However, in view of the symmetry of (a_{ij}), we have

$$\begin{aligned}(B[v], v_t) &= -\sum_{i,j=1}^{n} (a_{ij}v_{x_i t}, v_{x_j})\\ &= -\sum_{i,j=1}^{n} \int_D [a_{ij}/2][v_{x_i}v_{x_j}]_t\, dx\, dt = 0.\end{aligned} \tag{2.5}$$

Similarly, we get

$$(B[w], w_t) = 0. \tag{2.6}$$

Equation (2.4) combined with (2.5) and (2.6) yields

$$\operatorname{Re}(\gamma B[z], z_t) = \beta(B[v], w_t) - \beta(B[w], v_t). \tag{2.7}$$

We also have

$$\begin{aligned}\operatorname{Re}(\gamma B[z], k\lambda^{-1}z) &= \alpha(B[v], k\lambda^{-1}v) + \alpha(B[w], k\lambda^{-1}w)\\ &\quad + \beta(B[v], k\lambda^{-1}w) - \beta(B[w], k\lambda^{-1}v).\end{aligned} \tag{2.8}$$

Since B is a symmetric operator, the last two terms in (2.8) add to zero. Thus (2.8) becomes

$$\operatorname{Re}(\gamma B[z], k\lambda^{-1}z) = \alpha(B[v], k\lambda^{-1}v) + \alpha(B[w], k\lambda^{-1}w). \tag{2.9}$$

With the use of (2.7) and (2.9), we get

$$\begin{aligned}&\|\gamma B[z]\|^2 + \|z_t\|^2 + \|k\lambda^{-1}z\|^2 - 2\operatorname{Re}(\gamma B[z], z_t + k\lambda^{-1}z)\\ &\quad = (\alpha^2 + \beta^2)\|B[z]\|^2 + \|z_t\|^2 + \|k\lambda^{-1}z\|^2\\ &\qquad -2\beta(B[v], w_t) + 2\beta(B[w], v_t) - 2\alpha(B[v], k\lambda^{-1}v)\\ &\qquad - 2\alpha(B[w], k\lambda^{-1}w)\\ &\quad = \alpha^2\|B[v]\|^2 - 2\alpha(B[v], k\lambda^{-1}v) + \|k\lambda^{-1}v\|^2\\ &\qquad + \beta^2\|B[v]\|^2 - 2\beta(B[v], w_t) + \|w_t\|^2\\ &\qquad + \alpha^2\|B[w]\|^2 - 2\alpha(B[w], k\lambda^{-1}w) + \|k\lambda^{-1}w\|^2\\ &\qquad + \beta^2\|B[w]\|^2 + 2\beta(B[w], v_t) + \|v_t\|^2\\ &\quad = \|\alpha B[v] - k\lambda^{-1}v\|^2 + \|\beta B[v] - w_t\|^2\\ &\qquad + \|\alpha B[w] - k\lambda^{-1}w\|^2 + \|\beta B[w] + v_t\|^2.\end{aligned} \tag{2.10}$$

Since the last term in (2.10) is nonnegative, this completes the proof.

Lemma 2. *If the operator L satisfies condition* (G), *$u \in P$, and k is any positive integer, then*

$$\|\lambda^{-k}L[u]\|^2 \geqslant k\|\lambda^{-k-1}u\|^2. \tag{2.11}$$

Proof. Let $z(x, t) = [\lambda(t)]^{-k}u(x, t)$. Then $z \in P$ and $L[u] = L[\lambda^k z] = \gamma\lambda^k B[z] - \lambda^k z_t - k\lambda^{k-1}z$. This identity yields

$$\begin{aligned}\|\lambda^{-k}L[u]\|^2 &= \|\gamma B[z] - z_t - k\lambda^{-1}z\|^2\\ &= \|\gamma B[z]\|^2 + \|z_t\|^2 + k^2\|\lambda^{-1}z\|^2\\ &\quad - 2\operatorname{Re}(\gamma B[z], z_t + k\lambda^{-1}z) + 2\operatorname{Re}(k\lambda^{-1}z, z_t).\end{aligned} \tag{2.12}$$

However, the last term in (2.12) may be written as

$$\begin{aligned} 2\mathrm{Re}(k\lambda^{-1}z, z_t) &= k\int_D \lambda^{-1}\left[|z|^2\right]_t dx\,dt = k\int \lambda^{-2}|z|^2 dx\,dt \\ &= k\|\lambda^{-1}z\|^2 = k\|\lambda^{-k-1}u\|^2. \end{aligned} \tag{2.13}$$

Combining (2.12) and (2.13), we have

$$\begin{aligned} &\|\lambda^{-k}L[u]\|^2 - k\|\lambda^{-k-1}u\|^2 \\ &\quad = \|\gamma B[z]\|^2 + \|z_t\|^2 + \|k\lambda^{-1}z\|^2 - 2\mathrm{Re}(\gamma B[z], z_t + k\lambda^{-1}z). \end{aligned} \tag{2.14}$$

From Lemma 1, we know that the right-hand side of equation (2.14) is nonnegative and this completes the proof.

THEOREM. *Suppose $u \in \tilde{P}$, the operator L satisfies condition* (G), *and*

$$\begin{aligned} &|L[u]|^2 \leqslant c|u|^2, \qquad (x, t) \in D, \\ &u(x, 0) = 0, \qquad x \in \Omega, \\ &u(s, t) = 0, \qquad (s, t) \in \partial\Omega \times [0, T]. \end{aligned} \tag{2.15}$$

Then $u(x, t) = 0$ for all $(x, t) \in D$.

PROOF. Suppose $t_1 \in (0, T)$. We wish to show $u(x, t) = 0$ for all $t \in [0, t_1]$, $x \in \Omega$. For this purpose choose $0 < t_1 < t_2 < t_3 < T$. Let ζ be a real-valued infinitely differentiable function defined on $[0, T]$ such that $\zeta(t) = 1$ for $t \in [0, t_2]$, $\zeta(t) = 0$ for $t \in [t_3, T]$, and $0 \leqslant \zeta(t) \leqslant 1$ for $t \in [t_2, t_3]$. Now set $v(x, t) = \zeta(t)u(x, t)$ and observe that $v \in P$. With the use of elementary methods and Lemma 2, we have

$$\begin{aligned} (t_2 + \eta)^{-2k}\int_{t_2}^{t_3}\int_\Omega |L[v]|^2 dx\,dt &\geqslant \int_{t_2}^{t_3}\int_\Omega \lambda^{-2k}|L[v]|^2 dx\,dt \\ &= \|\lambda^{-k}L[v]\|^2 - \int_0^{t_2}\int_\Omega \lambda^{-2k}|L[v]|^2 dx\,dt \\ &\geqslant k\|\lambda^{-k-1}v\|^2 - c\int_0^{t_2}\int_\Omega \lambda^{-2k}|v|^2 dx\,dt \\ &\geqslant k\int_0^{t_2}\int_\Omega \lambda^{-2k-2}|v|^2 dx\,dt - c\eta^{-2}\int_0^{t_2}\int_\Omega \lambda^{-2k-2}|v|^2 dx\,dt \\ &= (k - c\eta^{-2})\int_0^{t_2}\int_\Omega \lambda^{-2k-2}|v|^2 dx\,dt. \end{aligned} \tag{2.16}$$

For $k > c\eta^{-2}$, we obtain

$$\begin{aligned} &(k - c\eta^{-2})\int_0^{t_2}\int_\Omega \lambda^{-2k-2}|v|^2 dx\,dt \\ &\qquad \geqslant (k - c\eta^{-2})(t_1 + \eta)^{-2k-2}\int_0^{t_1}\int_\Omega |v|^2 dx\,dt. \end{aligned} \tag{2.17}$$

Combining (2.16) and (2.17) and simplifying, we get

$$[\eta^2/(\eta^2 k - c)][(t_1+\eta)/(t_2+\eta)]^{2k}\int_{t_2}^{t_3}\int_\Omega |L[v]|^2 dx\,dt \tag{2.18}$$
$$\geqslant (t_1+\eta)^{-2}\int_0^{t_1}\int_\Omega |v|^2 dx\,dt.$$

Since $t_1 < t_2$, the left-hand side of (2.18) approaches zero as $k \to \infty$ and this implies the right-hand side is zero since it is independent of k. This completes the proof.

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