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DILATIONS ON INVOLUTION SEMIGROUPS[1]

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ABSTRACT. We present an equivalent form of the boundedness condition involved in the Sz.-Nagy general dilation theorem and, as a consequence, we prove a dilation theorem for a product of commuting dilatable operator functions on involution semigroups. Also we show that the Bram-Halmos criterion of subnormality can be directly deduced from the proposed boundedness condition.

Let H be a Hilbert space and let S be an involution semigroup (shortly a $*$-semigroup). Assume S has a unit. Denote by $F(S, H)$ the set of all functions from H to S with finite supports. We will denote by f and g elements of $F(S, H)$ as well as of H. In practice this ambiguity will cause no confusion.

Given an operator function $\varphi: S \to L(H)$ ($L(H)$ stands for the algebra of all bounded linear operators in H) we will say that $\varphi: S \to L(H)$ is positive definite (shortly p.d.) if $\sum_{s,t}(\varphi(s^*t)f(t), f(s)) \geqslant 0$, $f \in F(S, H)$. Recall that the function φ is said to be dilatable if there exist: a Hilbert space K, an involution preserving semigroup homomorphism Φ: $S \to L(K)$ and a bounded linear operator V: $H \to K$, such that $\varphi(s) = V^*\Phi(s)V$, $s \in S$. The general dilation theorem of Sz.-Nagy [4, Principal Theorem] says that an operator function φ: $S \to L(H)$ is dilatable if and only if it is p.d. and satisfies the following boundedness condition

$$\text{(BC)} \qquad \sum_{s,t} (\varphi(s^*u^*ut)f(t), f(s)) \leqslant c(u) \sum_{s,t} (\varphi(s^*t)f(t), f(s)), \qquad f \in F(S, H),$$

where $c(u)$ does not depend on $f \in F(S, H)$.

There are two cases when (BC) is needless: (a) S is a group ($s^* = s^{-1}$). Then (BC) becomes trivial. (b) S is a Banach star algebra. Then (BC) is a consequence of positive definiteness of φ, see [2]. In case of a general $*$-semigroup, the boundedness condition (BC) is still needed. In this note we offer an alternative boundedness condition (Proposition, condition (ii)) which in some cases can be more convenient. As applications we prove the Bram [1]

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improvement of Halmos' criterion of subnormality and also a dilation theorem for a product of commuting dilatable operator functions on involution semigroups.

PROPOSITION. *Suppose* $\varphi: S \to L(H)$ *is p.d. Then the following conditions are equivalent*:

(i) φ *satisfies* (BC);

(ii) $\|\varphi(s)\| \leqslant C\alpha(s)$, $s \in S$, *where* $\alpha(st) \leqslant \alpha(s)\alpha(t)$, $t, s \in S$, *and* $\alpha(1) = 1$;

(iii) $\liminf_{k\to\infty}(\sum_{s,t}(\varphi(s^*(u^*u)^{2^k}t)f(t), f(s)))^{2^{-k}} \leqslant d(u)$ *where* $d(u)$ *is independent of* $f \in F(S, H)$.

Although the implication (i) $\Rightarrow$ (ii) follows immediately from the dilation theorem, we would rather prove it directly in order to make the Proposition independent of any dilation argument. The proof of (i) $\Rightarrow$ (ii) goes as follows: (BC) implies $(\varphi(s^*s)f, f) \leqslant c(u)(\varphi(1)f, f)$, $f \in H$. The Schwarz inequality for φ (see [2]) implies $|(\varphi(1 \cdot s)f, g)|^2 \leqslant (\varphi(s^*s)f, f)(\varphi(1)g, g)$, $f, g \in H$. Putting together both these inequalities we infer that φ satisfies (ii) with $\alpha(s) = c(s)^{1/2}$ and $C = \|\varphi(1)\|$. Choose $c(u)$ to be minimal in (BC). The double use of (BC) implies $c(uv) \leqslant c(u)c(v)$ which proves the submultiplicativity of α. The verification of (ii) $\Rightarrow$ (iii), is standard due to the submultiplicativity of α. The proof of (iii) $\Rightarrow$ (i) is a substantial part of [3].

REMARK. The c, α and d involved in the Proposition can be related as follows: $c(u)$ minimal in (BC) $\Rightarrow \alpha(s) = c(s)^{1/2}$; $\alpha(s)$ as in (ii) $\Rightarrow d(u) \leqslant \alpha(u^*u)$; $d(u)$ as in (iii) $\Rightarrow c(u) \leqslant d(u)$.

Now we present some examples which point out usefulness of the Proposition.

1. A p.d. φ is norm bounded. Then, by (ii), it is dilatable (Arveson; for more detail see [3]).

2. Given $A \in L(H)$ such that

$$(*) \qquad \sum_{i,j} \left(A^i f_j, A^j f_i\right) \geqslant 0, \qquad f_1, \ldots, f_n \in H.$$

Following Sz.-Nagy we define S = the set of all pairs of nonnegative integers (i, j), $(i, j)(k, l) = (i + k, j + l)$, $(ij)^* = (j, i)$ and $\varphi(i, j) = A^{*j}A^i$. Then $(*)$ implies [4, §10] that the function φ is p.d. on the *-semigroup S. Moreover, $\|\varphi(i, j)\| \leqslant \|A\|^{i+j}$. This proves by the Proposition that φ satisfies (BC). Fix $f_1, \ldots, f_n$ in H. Set $f(i, 0) = f_i$, $i = 1, \ldots, n$, $f(k, l) = 0$ otherwise and $u = (1, 0)$. Then, invoking the Remark, (BC) can be written as

$$\sum_{i,j} \left(A^{i+1} f_j, A^{j+1} f_i\right) \leqslant \|A\|^2 \sum_{i,j} \left(A^i f_j, A^j f_i\right).$$

This proves Bram's theorem [1, Theorem 1] in an alternative way.

3. Given two dilatable operator functions $\varphi: S \to L(H)$ and $\psi: T \to L(H)$ such that

$$(**) \qquad \varphi(s)\psi(t) = \psi(t)\varphi(s), \qquad s \in S, t \in T.$$

Define $\chi(s, t) = \varphi(s)\psi(t)$ and consider the operator function χ on the

$*$-semigroup $S \times T$ where multiplication and involution are defined in coordinatewise fashion. Then χ is p.d. To see this fix f in $F(S \times T, H)$, say, $f(s_i, t_j) = f_{ij}$ and $f(s, t) = 0$ otherwise. By $(**)$ the operator matrix $(\psi(t_j^* t_l))$ commutes with $(\varphi(s_i^* s_k))$ and so does its square root. The latter decomposes as an $n \times n$ operator matrix, say, (A_{pq}), that is, $\psi(t_j^* t_l) = \sum A_{jr} A_{rl}$. Moreover, each A_{pq} commutes with each $\varphi(s_i^* s_k)$ (for i, k fixed the matrix $\operatorname{diag}(\varphi(s_i^* s_k))$ commutes with $(\psi(t_j^* t_l))$ and, consequently, so does it with (A_{pq})) and $A_{pq} = A_{qp}^*$. Thus we have

$$\sum_{i,j,k,l} \left(\varphi(s_i^* s_k)\psi(t_j^* t_l) f_{kl}, f_{ij}\right) = \sum_{i,j,k,l} \sum_{p} \left(\varphi(s_i^* s_k) A_{jp} A_{pl} f_{kl}, f_{ij}\right)$$

$$= \sum_{p} \sum_{i,k} \left(\varphi(s_i^* s_k) \sum_{l} A_{pl} f_{kl}, \sum_{j} A_{pj} f_{ij}\right) \geqslant 0$$

because of positive definiteness of φ.

Both φ and ψ satisfy (ii) and so does χ. We get right away the following:

THEOREM. *If φ and ψ are dilatable and commute, then χ is dilatable too. Moreover, if $\varphi(1) = \psi(1) =$ the identity operator in H, then $\varphi(s)\psi(t)f = P\Phi(s)\Psi(t)f$, $f \in H$, $s \in S$, $t \in T$ where P is the orthogonal projection of a suitable K on H and Φ and Ψ are dilations of φ and ψ respectively, which commute.*

The last statement follows from the fact that $(s, 1)(1, t) = (1, t)(s, 1)$. Then the dilation of χ factors as $\Phi(\cdot)\Psi(-)$ with Φ and Ψ as above. The Theorem generalizes the result of [5, Lemma] to $*$-semigroup context. Furthermore, we can extend this result to an arbitrary family of commuting operator functions on $*$-semigroups and obtain a generalization of [4, Theorem V].

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ADDED IN PROOF. Recently Professor C. Ryll-Nardzewski discovered a simple example of a p.d. function on a $*$-semigroup not satisfying (BC).

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