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FIXED POINT THEOREMS FOR MAPPINGS OF NONEXPANSIVE TYPE

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ABSTRACT. Constructive fixed point theorems for single-valued and compact-valued nonexpansive mappings, which map a closed convex subset C of a Banach space X into X and send the boundary of C relative to X into C, are given. Mappings for which the method of asymptotic center applies are also considered.

In this note we consider fixed point theorems for mappings $f\colon C \to X$ of the following types:

(1) f is single-valued, nonexpansive and $f(\partial C) \subset C$.

(2) f is multivalued, nonexpansive and $f(x) \subset C$ for $x \in \partial C$.

(3) f is single-valued and satisfies the condition

$$\limsup_m \limsup_n \|f^m(x) - f^n(y)\| \leqslant k \limsup_n \|x - f^n(y)\|$$

for some $k > 0$. In the sequel, unless otherwise stated, X denotes a uniformly convex Banach space, C a closed convex nonempty bounded subset of X and ∂C the boundary of C relative to X. If $f\colon C \to X$ is a mapping such that $f(\partial C) \subset C$, we define $F\colon C \to C$ by putting

$$F(x) = f(x) \quad \text{if } f(x) \in C,$$
$$= \text{the point where the line segment } [x, f(x)] \text{ and } \partial C \text{ intersect} \quad \text{if } f(x) \notin C.$$

If f is a contraction, it is known that f has a fixed point (Assad and Kirk [1]). If f is nonexpansive, by considering the contractions

$$f_\lambda(x) = \lambda x_0 + (1 - \lambda) f(x), \qquad 0 < \lambda < 1,\ x_0 \in C,$$

we get for each λ, a fixed point x_λ of f_λ, and it follows that $\|x_\lambda - f(x_\lambda)\| \to 0$ as $\lambda \to 0$.

The asymptotic center w.r.t. C of a bounded sequence $\{x_n\}$ in X is the unique point in C at which the function

$$r(x) = \limsup_n \|x - x_n\|, \qquad x \in C,$$

attains its minimum (Edelstein [2]). $r(x)$ is clearly a convex function. We begin with the following

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Proposition 1. *Suppose that the asymptotic center c w.r.t. C of $\{x_n\}$ is an interior point of C. Then c is also the asymptotic center of $\{x_n\}$ w.r.t. X.*

Proof. If not, let c' be the asymptotic center of $\{x_n\}$ w.r.t. X. Since c is an interior point of C, the line segment $[c, c']$ has a nontrivial intersection with C, i.e. there exists $0 < \lambda < 1$ such that $c_1 = \lambda c + (1 - \lambda)c' \in C$. Since $r(c') \leqslant r(c)$, it follows from the convexity of $r(x)$ that $r(c_1) \leqslant r(c)$. This contradicts the uniqueness of the asymptotic center.

For nonexpansive mappings existence of fixed points of f was proved by Kirk [6] under a more general setting. In case $f\colon C \to C$, part (a) of the following theorem was proved by Edelstein [2].

Theorem 1. *Let $f\colon C \to X$ be a nonexpansive mapping with $f(\partial C) \subset C$.*

(a) *The asymptotic center of the sequence $\{x, F(x), F^2(x), \ldots\}$ w.r.t. C is a fixed point of f.*

(b) *If $x_n - f(x_n) \to 0$, then the asymptotic center of $\{x_n\}$ w.r.t. C is a fixed point of f.*

Proof. Let c be the asymptotic center of $\{x, F(x), F^2(x), \ldots\}$ w.r.t. C. For each n, consider $\|f(c) - F^n(x)\|$.

Case (i). $f(F^{n-1}(x)) \in C$. Then $F^n(x) = f(F^{n-1}(x))$ and

$$\|f(c) - F^n(x)\| \leqslant \|c - F^{n-1}(x)\|. \tag{1}$$

Case (ii). $f(F^{n-1}(x)) \not\in C$. Then $F^{n-1}(x) = f(F^{n-2}(x))$ for, otherwise, $F^{n-1}(x) \in \partial C$ and hence $f(F^{n-1}(x)) \in C$ by hypothesis, a contradiction. Thus

$$\begin{aligned} F^n(x) &= \lambda F^{n-1}(x) + (1 - \lambda) f\big(F^{n-1}(x)\big) \\ &= \lambda f\big(F^{n-2}(x)\big) + (1 - \lambda) f\big(F^{n-1}(x)\big) \end{aligned}$$

and

$$\|f(c) - F^n(x)\| \leqslant \lambda\|F^{n-2}(x) - c\| + (1 - \lambda)\|F^{n-1}(x) - c\|. \tag{2}$$

It follows from (1) and (2) that

$$\limsup_n \|f(c) - F^n(x)\| \leqslant \limsup_n \|c - F^n(x)\|. \tag{3}$$

If $f(c) \in C$, then $f(c) = c$ by the uniqueness of asymptotic center. Otherwise $f(c) \not\in C$ and c must be an interior point of C. By Proposition 1, c is also the asymptotic center of $\{F^n(x)\}$ w.r.t. X, and hence by (1) and the uniqueness, $f(c) = c$, contradicting $f(c) \not\in C$. Hence $f(c) \in C$ and $f(c) = c$.

For part (b), let c be the asymptotic center of $\{x_n\}$ w.r.t. C. Then

$$\begin{aligned} \|f(c) - x_n\| &\leqslant \|f(c) - f(x_n)\| + \|f(x_n) - x_n\| \\ &\leqslant \|c - x_n\| + \|f(x_n) - x_n\|. \end{aligned}$$

Taking lim sup we have, $\limsup \|f(c) - x_n\| \leqslant \limsup \|c - x_n\|$.

The following theorem slightly generalizes Theorem 1 in [9].

Theorem 2. *Denote by $\mathcal{C}(X)$ the family of compact nonempty subsets of X*

equipped with the Hausdorff metric. If $f\colon C \to \mathcal{C}(X)$ *is nonexpansive and* $f(x) \subset C$ *for all* $x \in \partial C$, *then* f *has a fixed point.*

PROOF. As in [9], there exist sequences $\{x_n\} \subset C$ and $\{y_n\} \subset X$ such that $y_n \in f(x_n)$, $\|x_n - y_n\| \to 0$ and all subsequences of $\{x_n\}$ have the same asymptotic center w.r.t C. Arguing as in [9], there exists $p \in f(c)$ such that $\limsup \|p - x_n\| \leqslant \limsup \|c - x_n\|$. If $f(c) \subset C$, then $p \in C$ and $p = c$ by the uniqueness of asymptotic center w.r.t. C; if not, c must be an interior point and, by Proposition 1, $p = c$. Hence $c \in f(c)$.

THEOREM 3. *Let* C *be a subset of a Banach space such that every bounded net in* C *has an asymptotic center, e.g. a closed convex subset of a reflexive Banach space. Let* Ω *be the first uncountable ordinal and* x_α, $\alpha < \Omega$, *be a transfinite sequence in* C *indexed by ordinals less than* Ω *such that* x_γ *is an asymptotic center* [8] *of* x_η, $\eta < \gamma$, *whenever* γ *is a limit ordinal. Define*

$$R_1(x_\gamma) = \sup\{\|x_\gamma - x_{\gamma+n}\|: n = 1, 2, \dots\},$$

$$R_2(x_\gamma) = \limsup \{\|x_\gamma - x_{\gamma+n}\|: n = 1, 2, \dots\},$$

$$R_3(x_\gamma) = \inf_{x \in C} \limsup\{\|x_\gamma - x_{\gamma+n}\|: n = 1, 2, \dots\}$$

and suppose $f\colon C \to C$ *is a function such that* $f(x_\gamma) = x_{\gamma+1}$. *If the following condition is satisfied for some* $i = 1, 2, 3$:

$$R_i(x_\delta) \leqslant R_i(x_\gamma), \qquad \delta > \gamma \quad \text{and} \quad R_i(x_\gamma) > R_i(x_{\gamma+\omega}) \quad \text{if } R_i(x_\gamma) > 0,$$

then (i) f *has a fixed point if* $i = 1$, (ii) *there exists a point* x *such that* $\lim_n f^n(x) = x$ *if* $i = 2$, *and* (iii) *there exists a point* x *such that* $\lim_n f^n(x)$ *exists if* $i = 3$.

PROOF. The transfinite sequence of nonnegative real numbers $R_i(x_\gamma)$, $\gamma < \Omega$ is decreasing and must be eventually constant. There exists a point x_γ such that $R_i(x_\gamma) = R_i(x_{\gamma+\omega})$ and hence $R_i(x_\gamma) = 0$. The conclusion in the theorem then follow immediately.

The proofs of the following lemmas were given in [10].

LEMMA 1. *Let* $\{x_n\}$ *be a bounded sequence in* X. *For each* $x \in X$ *let* $r(x) = \limsup_n \|x - x_n\|$. *Let* $\delta(\varepsilon)$ *be the modulus of convexity of* X.

(i) *If* $r(x) \leqslant d$, $r(y) \leqslant d$ *and* $\|x - y\| \geqslant \varepsilon$, *then* $r((x + y)/2) \leqslant d(1 - \delta(\varepsilon/d))$ *(or* $\delta(\varepsilon/d -)$ *if* $\varepsilon/d = 2$).

(ii) $|r(x) - r(y)| \leqslant \|x - y\| \leqslant r(x) + r(y)$.

LEMMA 2. *Suppose that in Lemma* 1, $\{x_n\}$ *is relatively compact and* C *is a closed convex subset of* X. *Then the asymptotic center of* $\{x_n\}$ *w.r.t.* C *is the Chebyshev center of the set of subsequential limits of* $\{x_n\}$ *w.r.t.* C.

The following two theorems are slight generalizations and modifications of Theorems 1 and 2 in Kirk [7] and Theorem 1 in Goebel and Kirk [4]. The first illustrates the method used in Theorem 3 with $R_3(x_{\omega^2}) = 0$. Let X be a

Banach space with $\delta(1) > 0$. X is reflexive (James [5]). Let $\alpha > 1$ be the solution of $\alpha(1 - \delta(1/\alpha)) = 1$.

THEOREM 4. *Let X be defined as above and C a closed convex subset of X. Let f: $C \to C$ be a mapping satisfying the following condition*:

(i) $$\limsup_m \limsup_n \|f^m(x) - f^n(y)\| \leqslant k \limsup_n \|x - f^n(y)\|,$$

where $0 < k < \alpha$. Then there exists an x with $\lim_n f^n(x) = x$. If f^N is continuous for some $N > 0$, then x is a fixed point of f.

PROOF. We can assume $1 < k < \alpha$. Choose k', $k < k' < \alpha$. Define x_γ for $r < \omega^2$ as in Theorem 3. Denote $y_n = x_{n\omega}$. For each n, let

$$r_n(x) = \limsup_m \|x - f^m(y_n)\|.$$

Thus $f_{n-1}(y_n) = R_3(y_{n-1})$ in the notation of Theorem 3. We shall write s_n for $r_{n-1}(y_n)$.

If $s_n = 0$ for some n, then $f^i(y_{n-1}) \to y_n$ and it follows from (i) that $f^i(y_n) \to y_n$. Therefore we assume $s_n > 0$.

CLAIM.

(ii) $$r_n(y_n) \leqslant (k'/\alpha) r_{n-1}(y_{n-1}).$$

Since $k' > 1$, we have $r_{n-1}(y_n) \leqslant k's_n$. For sufficiently large m, we have, from (i), $r_{n-1}(f^m(y_n)) \leqslant k's_n$. If $\|y_n - f^m(y_n)\| > (k'/\alpha)s_n$, then by Lemma 1

$$r_{n-1}\left(\frac{y_n + f^m(y_n)}{2}\right) \leqslant k's_n\left(1 - \delta\left(\frac{1}{\alpha}\right)\right) = \frac{k'}{\alpha} s_n < s_n = R_3(y_{n-1}),$$

a contradiction to the definition of y_n. Hence we have $\|y_n - f^m(y_n)\| \leqslant (k'/\alpha)s_n$ for sufficiently large m, which implies

$$r_n(y_n) \leqslant (k'/\alpha)s_n \leqslant (k'/\alpha) r_{n-1}(y_{n-1}).$$

From (ii) it follows that $r_n(y_n) \leqslant (k'/\alpha)^{n-1} r_1(y_1)$.

Since

$$\|y_{n+1} - y_n\| \leqslant r_{n+1}(y_{n+1}) + r_{n+1}(y_n) \leqslant 2r_{n+1}(y_{n+1}) \leqslant 2(k'/\alpha)^n r_1(y_1),$$

the sequence y_n is a Cauchy sequence. Let z denote its limit. It follows from the inequalities (from Lemma 1)

$r_n(z) \leqslant r_n(y_n) + \|y_n - z\|$,

$\|f^i(z) - z\| \leqslant r_n(f^i(z)) + r_n(z)$, and

$\limsup_i r_n(f^i(z)) \leqslant k r_n(z)$ (from (i)) that $r_n(z) \to 0$ and $f^i(z) \to z$.

If f^N is continuous for some N, then $f(z) = \lim_p f^{Np+1}(z) = z$.

THEOREM 5. *Let K be a nonempty compact convex subset of a Banach space X. Let f: $K \to K$ be a mapping satisfying*

(A) $$\limsup_m \limsup_n \|f^m(x) - f^n(y)\| \leqslant \limsup_n \|x - f^n(y)\|.$$

Then there exists a point x such that $\lim_n f^n(x) = x$. *If* f^N *is continuous for some* $N > 0$, *then* $f(x) = x$.

PROOF. Let M be a subset of K minimal w.r.t. being nonempty, closed, convex and that for every $x \in M$, every subsequential limit of $\{f^n(x)\}$ is in M. Such an M exists by a standard argument of Zorn's Lemma. Suppose that M consists of more than one point. Let $x_0 \in M$ and let C be the asymptotic center of $\{f^n(x_0)\}$ w.r.t. M. By Lemma 2, C is the Chebyshev center w.r.t. M of the set D of subsequential limits of $\{f^n(x_0)\}$ in M. By hypothesis $D \subset M$. Since M has normal structure [11], C is a proper subset of M. For each $x \in K$, let $r(x) = \limsup_n \|x - f^n(x_0)\|$ and $r_1 = \inf\{r(x): x \in M\}$. Then $C = \{x \in M: r(x) = r_1\}$. If $y \in C$, then it follows from condition (A) that $\limsup_i r(f^i(y)) \leqslant r(y)$. Let $\{f^{n_i}(y)\}$ be a convergent subsequence of $\{f^n(y)\}$ with limit z. By continuity of r, we have

$$\limsup_i r(f^{n_i}(y)) = r(z) \leqslant \limsup_n r(f^n(y)) \leqslant r(y) = r_1.$$

Since $z \in M$ we must have $r(z) = r_1$, i.e. $z \in C$. This shows that C is a nonempty closed convex proper subset of M and every subsequential limit of $\{f^n(x)\}$ is in C for every $x \in C$, a contradiction to the minimality of M. Hence M consists of one point z_0. Obviously we have $\lim f^n(z_0) = z_0$. The last part of the theorem follows from the proof of Theorem 4.

Finally we remark that inequality (i) in Theorem 4 is implied by each of the following:

(i) $\qquad \|f^i(x) - f^i(y)\| \leqslant k_i\|x - y\|, \quad i \geqslant N, \quad k_i \to 1 \quad [3],$

(ii) $\qquad \|f^i(x) - f^i(y)\| \leqslant k\|x - y\|, \quad i \geqslant N \quad [4],$

(iii) $\qquad \limsup_i \sup_{y \in C} \{\|f^i(x) - f^i(y)\| - \|x - y\|\} \leqslant 0 \quad [7].$

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