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A CLASS OF ONE-PARAMETER NONLINEAR SEMIGROUPS WITH DIFFERENTIABLE APPROXIMATING SEMIGROUPS

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ABSTRACT. Suppose that T is a strongly continuous semigroup of transformations on a subset C of a Banach space X. For $\delta > 0$, consider $U_\delta(t) = \{(\delta^{-1}\int_0^\delta g_x, \delta^{-1}\int_0^\delta g_{T(t)x}): x \in C\}$ where g_x denotes the trajectory of T from x. The class H of semigroups for which $U_\delta(t)$ is a function for $\delta > 0$ and $t \geqslant 0$ contains all strongly continuous linear semigroups and Webb's nonlinear nonexpansive example with no dense set of differentiability. If $T \in H$, $U_\delta = \{(t, U_\delta(t)): t \geqslant 0\}$ is a semigroup on $\{\delta^{-1}\int_0^\delta g_x: x \in C\}$ with continuously differentiable trajectories. Also, as $\{\delta_n\}_{n=1}^\infty$ converges to 0, the trajectories of $\{U_{\delta_n}\}_{n=1}^\infty$ uniformly approximate the trajectories of T.

1. **Introduction.** This paper deals with the construction of differentiable semigroups which uniformly approximate a given strongly continuous semigroup. In the Hille-Yosida theory for strongly continuous semigroups of continuous linear transformations, points of the form $\delta^{-1}\int_0^\delta g_x$, $\delta > 0$, are known to have the property that if p is one of them, then g_p, the trajectory of the semigroup from p, is differentiable. Thus linear semigroups have differentiable trajectories from a dense set of points. That this dense differentiability need not occur in nonlinear semigroups on Banach spaces was first shown in an example due to Webb (see [4]). Even lacking differentiability, however, Brezis and Pazy [1] and Crandall and Liggett [2] have shown that if a semigroup T of local type is given by an exponential formula

$$T(t)x = \lim_{n\to\infty} (I + (t/n)A)^{-n}x,$$

where A is computed from $(I - (t/\delta)(T(\delta) - I))^{-1}$, then the trajectories of T are uniformly approximated by the semigroups whose trajectories are solutions to

$$y'(t) = \delta^{-1}(T(\delta) - I)(y(t)).$$

The theorem to follow gives a different approximation technique not dependent upon the existence of an exponential generator. Define, for a given strongly continuous semigroup T on a subset C of a Banach space X, a relation $U_\delta(t)$, $\delta > 0$, $t \geqslant 0$, so that $U_\delta(t)(\delta^{-1}\int_0^\delta g_x) = \delta^{-1}\int_0^\delta g_{T(t)x}$ for each point x of C. The collection H of semigroups for which $U_\delta(t)$ is a function for all positive δ and nonnegative t includes Webb's example and all strongly

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continuous linear semigroups. Note that T is in class H if $x \neq y$ implies $\int_0^\delta g_x \neq \int_0^\delta g_y$. Given T in H, $\delta > 0$, and $A_\delta = \{(\delta^{-1}\int_0^\delta g_x, \delta^{-1}(T(\delta) - I)x): x \in C\}$, $\{(t, U_\delta(t)): t \geq 0\}$ is a semigroup on $\{\delta^{-1}\int_0^\delta g_x: x \in C\}$ with differentiable trajectories satisfying $y'(t) = A_\delta(y(t))$.

Conditions for a semigroup on a subset of $C_{[0,1]}$ are given which guarantee its inclusion in H and examples are presented which may indicate what type of semigroups may be expected to belong to H.

2. Definitions and theorems.

Notation. In the following definitions and theorems, X denotes a Banach space, and C denotes a subset of X. I denotes the identity function on a given subset of X and j denotes the identity function on the numbers.

DEFINITION. The statement that T is a semigroup on C means that (i) $T: [0, \infty) \to \{f: f: C \to C\}$, (ii) $T(0) = I$, and (iii) if t_1 and t_2 are nonnegative numbers

$$T(t_1 + t_2) = T(t_1)(T(t_2)).$$

DEFINITION. The statement that g_x is a trajectory of the semigroup T on C from the point x means that $g_x = \{(t, T(t)x): t \in [0, \infty)\}$.

DEFINITION. The statement that the semigroup T is strongly continuous means that each trajectory of T is continuous.

DEFINITION. Suppose that T is a strongly continuous semigroup on C. The statement that T is in class H means that if $\delta > 0$ and $t \geq 0$, then $\{(\delta^{-1}\int_0^\delta g_x, \delta^{-1}\int_0^\delta g_{T(t)x}): x \in C\}$ is a function, that is, is single-valued.

THEOREM 1. *Suppose that T is a strongly continuous semigroup on C and T is in class H. For $\delta > 0$, define $C_\delta = \{\delta^{-1}\int_0^\delta g_x: x \in C\}$. For $t \geq 0$, define $U_\delta(t): C_\delta \to C_\delta$ by*

$$U_\delta(t)\left(\delta^{-1}\int_0^\delta g_x\right) = \delta^{-1}\int_0^\delta g_{T(t)x}$$

and $A_\delta: C_\delta \to X$ by

$$A_\delta\left(\delta^{-1}\int_0^\delta g_x\right) = \delta^{-1}((T(\delta) - I)x).$$

Then (i) $\{(t, U_\delta(t)): t \geq 0\}$ *is a semigroup on C_δ with continuously differentiable trajectories satisfying $y'(t) = A_\delta(y(t))$, and* (ii) *if $x \in C$ and $\{\delta_n\}_{n=1}^\infty$ is a sequence of positive numbers converging to* 0, *then if y_n is the trajectory of U_{δ_n} from $\delta_n^{-1}\int_0^{\delta_n} b_x$, $\{y_n\}_{n=1}^\infty$ converges uniformly to g_x on every interval in* $[0, \infty)$.

THEOREM 2. *Suppose that T is a strongly continuous semigroup on the subset C of $C_{[0,1]}$ so that if f and g are points of C and $f(x) > g(x)$, then*

$$[T(t)f](x) \geq [T(t)g](x)$$

for each number t in $[0, \infty)$.

Then $T \in H$.

3. **Proofs of the theorems.**

Theorem 1. Denote by S the strongly continuous semigroup of nonexpansive linear transformations on the Banach space Y of bounded uniformly continuous functions from $[0, \infty)$ into X (with lub norm) whose action at a point f of Y is $S(t)f = f(j + t)$. By definition of j, $[S(t)f](\delta) = f(j + t)(\delta) = f(\delta + t)$. (The use of a linear semigroup to study nonlinear semigroups was suggested by the work of Neuberger [3].) From the Hille-Yosida theory of strongly continuous linear semigroups, if F_f is the trajectory of S from $f \in Y$ and $\delta > 0$, the trajectory from $\delta^{-1}\int_0^\delta F_f$ is differentiable. Furthermore, if A is defined by

$$A\left(\delta^{-1}\int_0^\delta F_f\right) = F'_{\delta^{-1}\int_0^\delta F_f}(0),$$

then the integral equation

$$(S(t) - I)\int_0^r F_f = (S(r) - I)\int_0^t F_f$$

implies that $A(\delta^{-1}\int_0^\delta F_f) = \delta^{-1}(S(\delta) - I)f$.

Let $\delta > 0$ and C_δ and $U_\delta(t)$ be as in the hypothesis. Then $U_\delta(t)$ is a function from C_δ to C_δ by assumption. $U_\delta(0) = \{(\delta^{-1}\int_0^\delta g_x, \delta^{-1}\int_0^\delta g_{T(0)x}) \colon x \in C\} = I$.

$$\begin{aligned} U_\delta(t + s)\left(\delta^{-1}\int_0^\delta g_x\right) &= \delta^{-1}\int_0^\delta g_{T(t+s)x} = \delta^{-1}\int_0^\delta g_{T(t)T(s)x} \\ &= U_\delta(t)\left(U_\delta(s)\left(\delta^{-1}\int_0^\delta g_x\right)\right). \end{aligned}$$

Thus U_δ is a semigroup on C_δ.

The strong continuity of U_δ is demonstrated by showing that a given trajectory is continuously differentiable. Suppose $t \geqslant 0$, h_x is the trajectory of U_δ from $\delta^{-1}\int_0^\delta g_x$, and f is an element of Y which agrees with g_x on $[0, \delta + t]$. Riemann sum approximations to $\delta^{-1}\int_0^\delta F_f$ converge uniformly to the function $\delta^{-1}\int_0^\delta F_f$ and thus converge pointwise as well. Thus

$$\begin{aligned} \left[\lim_{n\to\infty} \sum_{k=1}^{n} n^{-1}f(j + (k\delta/n))\right](t) &= \lim_{n\to\infty} \sum_{k=1}^{n} n^{-1}f(t + (k\delta/n)) \\ &= \left[\delta^{-1}\int_0^\delta F_f\right](t). \end{aligned}$$

Since f agrees with g_x on $[0, \delta + t]$,

$$\begin{aligned} \sum_{k=1}^{n} n^{-1}f(t + (k\delta/n)) &= \sum_{k=1}^{n} n^{-1}g_x(t + (k\delta/n)) \\ &= \sum_{k=1}^{n_{n-1}} n^{-1}T(t + (k\delta/n))x \\ &= \sum_{k=1}^{n} n^{-1}T((k\delta)/n)T(t)x. \end{aligned}$$

Thus $[\delta^{-1}\int_0^\delta F_f](t) = \delta^{-1}\int_0^\delta g_{T(t)x} = U_\delta(t)(\delta^{-1}\int_0^\delta g_x) = h_x(t)$ and h_x on any interval is a domain element of A on that interval and is therefore continuously differentiable. Furthermore,

$$\begin{aligned} h_x'(t) &= A\left(\delta^{-1}\int_0^\delta F_f\right)(t) = \left[\delta^{-1}(S(\delta) - I)f\right](t) \\ &= \delta^{-1}(f(j+\delta) - f)(t) = \delta^{-1}(g_x(t+\delta) - g_x(t)) \\ &= \delta^{-1}(T(\delta) - I)(T(t)x). \end{aligned}$$

Therefore, $h_x'(t) = A_\delta(h_x(t))$.

Suppose $x \in C$ and $\{\delta_n\}_{n=1}^\infty$ is a sequence of positive numbers converging to 0. Then $\{\delta_n^{-1}\int_0^{\delta_n} F_f\}_{n=1}^\infty$ converges uniformly to f. Since f agrees with g_x on $[0, t]$, the trajectories of U_{δ_n} from $\delta_n^{-1}\int_0^{\delta_n} g_x$ which are the restrictions of $\delta_n^{-1}\int_0^{\delta_n} F_f$ to $[0, t]$ converge uniformly to g_x on $[0, t]$.

Proof of Theorem 2.

Lemma. *Suppose that T is a strongly continuous semigroup on the subset C of $C_{[0,1]}$. For $f \in C$, define a collection of functions $\{f^x\}_{x\in[0,1]}$ from $[0, \infty) \to \mathbf{R}$ by $f^x(t) = [T(t)f](x)$. Suppose further that if $\delta > 0$ and f and h are points of C, then there is $x \in [0, 1]$ so that $\delta^{-1}\int_0^\delta f^x \neq \delta^{-1}\int_0^\delta h^x$. Then $T \in H$.*

Suppose the hypothesis holds and f and h are in C. Then

$$\left[\delta^{-1}\int_0^\delta g_f\right](x) = \delta^{-1}\int_0^\delta f^x \quad \text{and} \quad \left[\delta^{-1}\int_0^\delta g_h\right](x) = \delta^{-1}\int_0^\delta h^x.$$

Since f and h are different, there is a number x where these integrals are different. Thus $f \neq h$ implies $\delta^{-1}\int_0^\delta g_f \neq \delta^{-1}\int_0^\delta g_h$ and $U_\delta(t)$ must be a function for any number $t \in [0, \infty)$. Thus $T \in H$.

Consider T as in the hypothesis to Theorem 2 and suppose $f \neq h$. Then there is $x \in [0, 1]$ so that $f(x) \neq h(x)$. Suppose $f(x) > h(x)$. Then if $t \geq 0$, $[T(t)f](x) \geq [T(t)h](x)$. But T is strongly continuous, so

$$\lim_{t\to 0}\left[T(t)f\right](x) = f(x) \quad \text{and} \quad \lim_{t\to 0}\left[T(t)h\right](x) = h(x).$$

Thus there is $\varepsilon > 0$ so that if $t \in [0, \varepsilon]$, $[T(t)f](x) > [T(t)h](x)$ and $\delta^{-1}\int_0^\delta f^x > \delta^{-1}\int_0^\delta h^x$. By the Lemma, $T \in H$.

4. **Examples.** (A) (G. F. Webb, see [4]). Let $X = C = C_{[0,1]}$. If $f \in C_{[0,1]}$, and $x \in [0, 1]$, define

$$\left[T(t)f\right](x) = \begin{cases} t + f(x) & \text{if } f(x) \geq 0, \\ t + (1/2)f(x) & \text{if } f(x) < 0 \text{ and } t + (1/2)f(x) \geq 0, \\ 2t + f(x) & \text{if } t + (1/2)f(x) < 0. \end{cases}$$

(B) (Crandall and Liggett [2]). Let $X = C_{[0,1]}$ and let C be the set to which $f \in X$ belongs provided that if $x \in [0, 1]$, $0 \leq f(x) \leq x$. Define

$$\left[T(t)f\right](x) = \min\{t + f(x), x\}.$$

(C) Let $X = C = C_{[0,1]}$. If $f \in X$, denote by f^* the maximum value of the function f and by f_* the minimum value of f. Define

$$[T(t)f](x) = \begin{cases} \min\{t + f(x), f^*\} & \text{if there is } x \text{ in } [0, 1] \\ & \text{so that } t + f(x) \leqslant f^*, \\ f_* + t & \text{if } f_* + t \geqslant f^*. \end{cases}$$

(A) and (B) are examples having the "order" property in Theorem 2. Example (C) fails to have this order property but satisfies the hypothesis to the Lemma to Theorem 2.

5. **Concluding remarks.** The approximation theorem in this paper seems to be different from those using resolvents. In general, the trajectories of U_δ are differentiable at all numbers. Furthermore, A_δ might be viewed as a perturbation of $\delta^{-1}(T(\delta) - I)$ in the sense of a shift in the domain rather than the range. The full extent to which semigroups given by resolvent formulas and those of class H are related is not known to the author. Two questions seem to be of special interest.

(i) Do there exist conditions on C which imply that a semigroup of nonexpansive transformations on C be in H?

(ii) Does there exist a semigroup of nonexpansive transformations given by a resolvent formula and defined on the closure of a convex open set which is not in H?

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