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A NOTE ON GENERATORS OF SEMIGROUPS

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ABSTRACT. The generator T of a norm-continuous semigroup of identity preserving positive linear mappings on a C^*-algebra $\mathfrak{A}$ is characterized as one that satisfies $T(u^*u) \geqslant u^*T(u) + T(u^*)u$ for all unitary elements u in $\mathfrak{A}$.

1. **Introduction.** Since 1948 the analytical theory of semigroups has made vigorous progress. In that year K. Yosida and E. Hille (see [2, p. 363]) independently characterized the infinitesimal generators for strongly continuous semigroups of contractions on a normed linear space as those densely defined closed linear operators T with $\|(\lambda I - T)^{-1}\| \leqslant \lambda^{-1}$ for all $\lambda > 0$. Later in 1960 G. Lumer and R. S. Phillips [5] found another description of those generators as those densely defined "dissipative" operators T with range $[I - T] = X$, where X is a semi-inner-product space. If the semigroup of contractions $\alpha(t)$ is norm-continuous, its infinitesimal generator becomes just bounded dissipative. In this note we are concerned with the norm-continuous semigroups of identity preserving positive linear mappings on a C^*-algebra and find a new characterization of their infinitesimal generators in terms of the C^*-algebra ordering. It would be more desirable to have a description of the generators of strongly continuous semigroups of positive (or completely positive) linear mappings, and that is currently being investigated by this author.

Lindblad's recent work [3] on generators of one-parameter semigroups of completely positive identity preserving linear mappings on a C^*-algebra showed an analogous result. As it is known that "positive" case does not follow automatically from "completely positive" case, one part of the proof of Theorem 1 is the same as that used in [3] and the other part is quite different.

2. **Preliminary.** Let X be a Banach space, X^* its dual space and $B(X)$ the Banach algebra of all bounded linear operators (or mappings) on X. Given a linear functional f on X and an element x in X, we define a linear functional (x, f) on $B(X)$ by $(x, f)(T) = f(T(x))$ for T in $B(X)$. It is easily seen that (i) $\|(x, f)\| \leqslant \|x\|\,\|f\|$, (ii) $(\alpha x, f) = \alpha(x, f)$ for scalar α, (iii) $(x_1 + x_2, f) =$

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$(x_1, f) + (x_2, f)$, (iv) $(x, f_1 + f_2) = (x, f_1) + (x, f_2)$.

The *spatial numerical range* $V(T)$ of a given linear mapping T in $B(X)$ is defined as

$$V(T) = \{(x, f)(T) | f(x) = 1 = \|f\| = \|x\|\}.$$

Let $\mathfrak{B}$ be a unital Banach algebra. Given f in $\mathfrak{B}^*$ and x in $\mathfrak{B}$, we define a linear functional on $\mathfrak{B}$ as $f_x(y) = f(yx)$ for all y in $\mathfrak{B}$ and observe (i) $\|f_x\| \leqslant \|f\| \, \|x\|$, (ii) $f_{\alpha x} = \alpha f_x$ for scalar α, (iii) $f_{x_1+x_2} = f_{x_1} + f_{x_2}$.

The *numerical range* $V(T, \mathfrak{B})$ of a given element T in $\mathfrak{B}$ is defined as

$$V(T, \mathfrak{B}) = \{f(T) | f(I) = 1 = \|f\|\},$$

where I is the identity element in $\mathfrak{B}$. It is known that $V(T, \mathfrak{B}) = \{f_x(T) | f(x) = 1 = \|f\| = \|x\|\}$. In the following we list several properties needed in this note whose proofs can be found in [1] (see Theorem 4, p. 84, Theorem 4, p. 30, Theorem 4, p. 28 and Theorem 6, p. 30) and [4].

PROPOSITION 1. $\overline{\mathrm{co}}\, V(T) = V(T, \mathfrak{B})$, *where* $\overline{\mathrm{co}}\, V(T)$ *is the closed convex hull of* $V(T)$. *As a consequence* $V(T, \mathfrak{B})$ *is always a closed convex subset in the complex plane* $\mathbf{C}$.

PROPOSITION 2.

$$\begin{aligned}\operatorname{Max}\{\operatorname{Re}\alpha;\ \alpha \in V(T, \mathfrak{B})\} &= \inf_{\alpha>0} \frac{1}{\alpha}\{\|I + \alpha T\| - 1\} \\ &= \lim_{\alpha\to 0^+} \frac{1}{\alpha}\{\|I + \alpha T\| - 1\}.\end{aligned}$$

PROPOSITION 3.

$$\begin{aligned}\operatorname{Max}\{\operatorname{Re}\alpha;\ \alpha \in V(T, \mathfrak{B})\} &= \sup_{\alpha>0} \frac{1}{\alpha}\{\log\|\exp(\alpha T)\|\} \\ &= \lim_{\alpha\to 0^+} \frac{1}{\alpha}\{\log\|\exp(\alpha T)\|\}.\end{aligned}$$

PROPOSITION 4. *Let* x *be in* $\mathfrak{B}$. *Then* $\operatorname{Re}\lambda \leqslant 0$ *for all* λ *in* $V(x, \mathfrak{B})$ *if and only if* $\|\exp(tx)\| \leqslant 1$ *for all* $t \geqslant 0$.

DEFINITION. An element x in $\mathfrak{B}$ is said to be *dissipative* if $\operatorname{Re}\lambda \leqslant 0$ for all λ in $V(x, \mathfrak{B})$.

A linear mapping T of a C^*-algebra $\mathfrak{A}$ into itself is called *selfadjoint* if $T(x^*) = T(x)^*$ for all x in $\mathfrak{A}$, and is called *positive* if $T(x)$ is positive for all positive elements x in $\mathfrak{A}$. It is easy to see that a positive linear mapping is also selfadjoint.

3. **The main theorem.** Let e^{tT} be a one-parameter norm-continuous semigroup of positive linear mappings on a C^*-algebra $\mathfrak{A}$ with $e^{tT}(1) = 1$, and T its generator (1 is the identity element in $\mathfrak{A}$, and all C^*-algebras considered in this note have identity element). From $T(x) = \lim_{t\to 0^+}(e^{tT}(x) - x)/t$ $(x \in \mathfrak{A})$, we see that T is selfadjoint and $T(1) = 0$. In the following is

the theorem in which the generator T is characterized by C^*-algebra ordering structure.

THEOREM 1. *Let T be a bounded selfadjoint linear mapping on a C^*-algebra $\mathfrak{A}$ with $T(1) = 0$. The following two conditions are equivalent*:

(i) *T is dissipative.*

(ii) $T(u^*)u + u^*T(u) \leqslant 0$ *for all unitary elements u in $\mathfrak{A}$.*

PROOF. (ii) $\Rightarrow$ (i). The argument used here is the same as that in Proposition 4 in [3]. For the sake of completeness we do it as below. Because of Proposition 2 and a theorem due to Russo and Dye [6], it suffices to show that

$$r(T) \equiv \lim_{t\to 0^+} \sup_u \frac{1}{t}\left\{\|u + tT(u)\| - 1\right\} \leqslant 0,$$

where the supremum is taken over all unitary elements in $\mathfrak{A}$.

$$\begin{aligned}\|u + tT(u)\|^2 &= \|u^*u + u^*tT(u) + tT(u^*)u + t^2T(u^*)T(u)\| \\ &\leqslant \|1 + t^2T(u^*)T(u)\| \leqslant 1 + t^2\|T\|^2.\end{aligned}$$

Hence, $\|I + tT\|^2 \leqslant 1 + t^2\|T\|^2$, and

$$(\|I + tT\| - 1)(\|I + tT\| + 1) \leqslant t^2\|T\|^2,$$

hence,

$$(\|I + tT\| - 1)/t \leqslant t\|T\|^2/(\|I + tT\| + 1).$$

Therefore $r(T) \leqslant 0$.

(i) $\Rightarrow$ (ii). By a theorem due to Russo and Dye [6] we have

$$\begin{aligned}\|I + tT\|^2 &= \sup_{\substack{u:\text{unitary}\\ \text{in } \mathfrak{A}}} \|(I + tT)(u)\|^2 \\ &= \sup_u \|(u + tT(u))^*(u + tT(u))\|.\end{aligned}$$

Hence, for any state ϕ on $\mathfrak{A}$ and unitary element u in $\mathfrak{A}$, we have

$$\phi\big((u + tT(u))^*(u + tT(u))\big) \leqslant \|I + tT\|^2,$$

$$\phi\big(u^*u + t(T(u^*)u + u^*T(u)) + t^2T(u^*)T(u)\big) \leqslant \|I + tT\|^2,$$

$$t\phi(T(u^*)u + u^*T(u)) + t^2\phi(T(u^*)T(u)) \leqslant \|I + tT\|^2 - 1,$$

$$\begin{aligned}&\phi(T(u^*)u + u^*T(u)) + t\phi(T(u^*)T(u)) \\ &\qquad \leqslant ((\|I + tT\| - 1)/t)(\|I + tT\| + 1).\end{aligned}$$

Taking the limit for both sides as $t \to 0^+$ we have

$$\phi(T(u^*)u + u^*T(u)) \leqslant r(T)\cdot 2 \leqslant 0$$

because $r(T) \leqslant 0$. Therefore $T(u^*)u + u^*T(u) \leqslant 0$. Q.E.D.

COROLLARY 1. *The generator T of a norm-continuous semigroup of identity preserving positive linear mappings on a C^*-algebra satisfies condition* (ii) *in Theorem* 1.

PROOF. It is because of Proposition 4 and Theorem 1. Q.E.D.

Without the assumption $T(1) = 0$ in the above corollary, condition (ii) can be replaced by

(ii)′ $T(1) + u^*T(1)u - T(u^*)u - u^*T(u) \geqslant 0$, for all unitary u in $\mathfrak{A}$.
Define $T'(x) = T(x) - \frac{1}{2}(T(1)x + xT(1))$ for all x in $\mathfrak{A}$. Since

$$T'(u^*)u + u^*T'(u) = T(u^*)u + u^*T(u) - u^*T(1)u - T(1),$$

condition (ii) holds for T' if and only if condition (ii)′ holds for T. Denote $x \to Kx + xK$ by $\{K, x\} \equiv T_K(x)$. The semigroup generated by T_K is $\exp tT_K(x) = e^{tK}xe^{tK}$. The semigroup generated by $T' + T''$ $(= T)$, where $T'' = T_{(1/2)T(1)}$, is given by the Lie-Trotter formula

$$\exp t(T' + T'') = \lim_{n\to\infty} \left[\exp(tT'/n)\exp(tT''/n)\right]^n.$$

Hence $\exp t(T' + T'')$ is positive if T satisfies (ii)′. Conversely, suppose that $\exp tT$ is positive. Then T is selfadjoint. By the Lie-Trotter formula the semigroup generated by $T - T''$ $(= T')$ is positive, and $(\exp tT')(1) = 1$ for all $t \geqslant 0$. By Corollary 1, condition (ii) holds for T'. Therefore condition (ii)′ holds for T. We have concluded

COROLLARY 2. *Let T be a bounded selfadjoint linear mapping from $\mathfrak{A}$ into itself. Then* $\exp tT$ *is positive for all $t \geqslant 0$ iff condition* (ii)′ *holds for T.*

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