PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 66, Number 2, October 1977

ON QUASI-UNIFORMITIES IN HYPERSPACES

GILLES BERTHIAUME

ABSTRACT. Quasi-uniformities in spaces of subsets are investigated. It is shown, by means of examples, how quasi-uniformities can play a unifying role in the study of topological structures in hyperspaces. The main application is the following result: an infinite product of upper semicontinuous multi-valued mappings, with compact sets as values, is again upper semicontinuous.

1. **Introduction.** If $(X, \mathfrak{T})$ is a topological space, we denote the *upper* (resp. *lower*) *semifinite topology* induced by $\mathfrak{T}$ on $\mathcal{P}(X)$ by $\mathfrak{T}^*$ (resp. $\mathfrak{T}_*$). The *finite* (Vietoris) *topology* will be denoted by $\mathfrak{T}^\sim$. In the appendix to his basic paper [6], E. Michael apologizes to the reader for originally not having been aware of the fact that $\mathfrak{T}^\sim$ can be expressed as the join of two hyperspace topologies, viz. $\mathfrak{T}^*$ and $\mathfrak{T}_*$. He also mentions that, if $(X, \mathfrak{U})$ is a uniform space, then the (Bourbaki) uniformity induced by $\mathfrak{U}$ on $\mathcal{P}(X)$ can also be expressed as the join of two uniformities.

In fact, as noted by N. Levine and W. J. Stager in [5], these two structures are only *quasi*-uniformities, i.e. they do not necessarily satisfy the symmetry axiom (for basic definitions and results on quasi-uniform spaces, see [7]). On the other hand, they can be defined even if $\mathfrak{U}$ itself is only assumed to be a quasi-uniformity. Indeed, as U runs through the set of entourages $\mathfrak{U}$, the sets

$$U^* = \{(A, B) \in \mathcal{P}(X) \times \mathcal{P}(X) : B \subset U(A)\}$$

form a base for a quasi-uniformity $\mathfrak{U}^*$ on $\mathcal{P}(X)$, while the sets

$$U_* = \{(A, B) \in \mathcal{P}(X) \times \mathcal{P}(X) : A \subset U^{-1}(B)\}$$

form a base for a quasi-uniformity $\mathfrak{U}_*$ on $\mathcal{P}(X)$. We call $\mathfrak{U}^*$ (resp. $\mathfrak{U}_*$) the *upper* (resp. *lower*) *quasi-uniformity* induced by $\mathfrak{U}$ on $\mathcal{P}(X)$. Finally, the sets $U^\sim = U^* \cap U_*$ form a base for a quasi-uniformity $\mathfrak{U}^\sim$ on $\mathcal{P}(X)$, which is the join of $\mathfrak{U}^*$ and $\mathfrak{U}_*$, and which we call the *Bourbaki quasi-uniformity* induced by $\mathfrak{U}$ on $\mathcal{P}(X)$. If $\mathfrak{U}$ is itself a uniformity, then $\mathfrak{U}^*$ and $\mathfrak{U}_*$ are conjugate quasi-uniformities, and $\mathfrak{U}^\sim$ is nothing but the usual Bourbaki uniformity.

The aim of this paper is to study these hyperspace quasi-uniformities. In §2, we investigate some of their basic properties, including the relationships between quasi-uniformities and topologies. One of our results can be paraphrased as follows: a precompact union of precompact sets is precom-

Received by the editors February 17, 1977 and, in revised form, April 12, 1977.
AMS (MOS) subject classifications (1970). Primary 54B20, 54C60; Secondary 54E15.
Key words and phrases. Hyperspace, quasi-uniformity, multi-valued mapping.

pact. The aim of §3 is to show, by means of two examples (one of them deals with separation, the other concerns the closure mapping), how quasi-uniformities can be used to give a common proof of different results about topological structures in hyperspaces. This is done with the use of Pervin's quasi-uniformity. We pursue that line of thought in §4, and show how this approach leads us to the following new result: an arbitrary (not necessarily finite!) product of upper semicontinuous multi-valued mappings, with compact sets as values, is again upper semicontinuous.

If X is a topological space, $\mathcal{F}(X)$ will denote the *set of all closed subsets* of X, and $\mathcal{K}(X)$ the *set of all compact subsets* of X. We also write $\mathcal{P}_0(X)$ (resp. $\mathcal{K}_0(X)$) for the set of all nonempty sets in $\mathcal{P}(X)$ (resp. $\mathcal{K}(X)$).

Standard references on hyperspaces are [3] and [6].

2. **Properties of quasi-uniformities in hyperspaces.** Perhaps the most important result of [6] concerning uniform spaces is that "a compact union of closed sets is closed" (Theorem 2.5.1, p. 157). The following generalization to quasi-uniform spaces can be proved in the same way.

THEOREM 2.1. *Let $(X, \mathcal{U})$ be a quasi-uniform space, and suppose that $\mathcal{Q}$ is a compact subset of $(\mathcal{P}(X), (\mathcal{U}^{-1})^*)$. If each set of $\mathcal{Q}$ is closed in $(X, \mathcal{U})$, then the union of the sets of $\mathcal{Q}$ is also closed.*

The following result, which is new even in the case of uniform spaces, deals with precompactness.

THEOREM 2.2. *Let $(X, \mathcal{U})$ be a quasi-uniform space, and suppose that $\mathcal{Q}$ is a precompact subset of $(\mathcal{P}(X), \mathcal{U}^*)$. If each set of $\mathcal{Q}$ is precompact in $(X, \mathcal{U})$, then the union of the sets of $\mathcal{Q}$ is also precompact.*

PROOF. Let A_0 be the union of the sets of $\mathcal{Q}$, and let U be any entourage in $\mathcal{U}$. Now there exists an entourage V in $\mathcal{U}$ with $V^2 \subset U$. Since $\mathcal{Q}$ is precompact, there exists a finite subset $\mathcal{B}$ of $\mathcal{Q}$ such that $\mathcal{Q} \subset V^*(\mathcal{B})$. Now each set B in $\mathcal{B}$ is precompact, and so there exists a finite subset F_B of B such that $B \subset V(F_B)$. Then, the union F of the family of sets $(F_B)_{B \in \mathcal{B}}$ is a finite subset of A_0. Moreover, we claim that $A_0 \subset U(F)$. Indeed, for each set A in $\mathcal{Q}$ there exists a set B in $\mathcal{B}$ such that $A \in V^*(B)$, and so

$$A \subset V(B) \subset V(V(F_B)) \subset V^2(F) \subset U(F).$$

Thus, $A_0 \subset U(F)$, showing that A_0 is precompact.

A multi-valued mapping Γ of a topological space $(X_1, \mathcal{T}_1)$ into a topological space $(X_2, \mathcal{T}_2)$ is well known to be *upper semicontinuous* (abbr. u.s.c.) if and only if the (single-valued) mapping $x \mapsto \Gamma(x)$ of $(X_1, \mathcal{T}_1)$ into $(\mathcal{P}(X_2), \mathcal{T}_2^*)$ is continuous. Similarly, we say that a multi-valued mapping Γ of a quasi-uniform space $(X_1, \mathcal{U}_1)$ into a quasi-uniform space $(X_2, \mathcal{U}_2)$ is *quasi-uniformly upper semicontinuous* (abbr. q.-u. u.s.c.) if the mapping $x \mapsto \Gamma(x)$ of $(X_1, \mathcal{U}_1)$ into $(\mathcal{P}(X_2), \mathcal{U}_2^*)$ is quasi-uniformly continuous (abbr. q.-u. continuous).

Since precompactness is preserved under quasi-uniform continuity, we obtain, as a direct consequence of Theorem 2.2:

COROLLARY 2.3. *Let Γ be a q.-u. u.s.c. multi-valued mapping of a quasi-uniform space $(X_1, \mathcal{U}_1)$ into a quasi-uniform space $(X_2, \mathcal{U}_2)$. If A is a precompact subset of $(X_1, \mathcal{U}_1)$, and if $\Gamma(x)$ is precompact in $(X_2, \mathcal{U}_2)$ for each $x \in A$, then $\Gamma(A)$ is also precompact.*

As far as the relations between the various hyperspace structures considered above are concerned, Lemma 3.2 of [6, p. 159] now takes the following form (if $(X, \mathcal{U})$ is a quasi-uniform space, we denote by $\mathcal{T}(\mathcal{U})$ the topology induced by $\mathcal{U}$ on X):

LEMMA 2.4. *Let $(X, \mathcal{U})$ be a quasi-uniform space.*

(a) *The topology $\mathcal{T}(\mathcal{U}^*)$ is coarser than $(\mathcal{T}(\mathcal{U}))^*$. Conversely, in order that every $(\mathcal{T}(\mathcal{U}))^*$- (resp. $(\mathcal{T}(\mathcal{U}))^{\sim}$-) neighbourhood of a point A_0 in $\mathcal{P}(X)$ contains a $\mathcal{T}(\mathcal{U}^*)$- (resp. $\mathcal{T}(\mathcal{U}^{\sim})$-) neighbourhood of A_0, it is necessary and sufficient that the sets $U(A_0)$ form a base of neighbourhoods of A_0 in X, as U runs through $\mathcal{U}$.*

(b) *The topology $\mathcal{T}(\mathcal{U}_*)$ is finer than $(\mathcal{T}(\mathcal{U}))_*$. Conversely, in order that every $\mathcal{T}(\mathcal{U}_*)$- (resp. $\mathcal{T}(\mathcal{U}^{\sim})$-) neighbourhood of a point A_0 in $\mathcal{P}(X)$ contains a $(\mathcal{T}(\mathcal{U}))_*$- (resp. $(\mathcal{T}(\mathcal{U}))^{\sim}$-) neighbourhood of A_0, it is necessary and sufficient that A_0 be a precompact subset of $(X, \mathcal{U}^{-1})$.*

Note that, if one further assumes X to be a T_1-space, then this result can also be formulated and proved for the space $\mathcal{F}(X)$, instead of $\mathcal{P}(X)$.

We draw the following conclusion from Lemma 2.4:

THEOREM 2.5.

(a) *The topology $\mathcal{T}(\mathcal{U}^*)$ coincides with $(\mathcal{T}(\mathcal{U}))^*$ on $\mathcal{K}(X)$.*

(b) *If $(X, \mathcal{U})$ is a uniform space, then, on $\mathcal{K}(X)$, the topologies $\mathcal{T}(\mathcal{U}_*)$ and $\mathcal{T}(\mathcal{U}^{\sim})$ also coincide with $(\mathcal{T}(\mathcal{U}))_*$ and $(\mathcal{T}(\mathcal{U}))^{\sim}$, respectively.*

Perhaps it is worth mentioning at this point that the Hausdorff metric on the space of all nonempty, closed, bounded subsets of a metric space can also be "split up" in two. More generally, if (X, d) is a quasi-pseudometric space (d does not necessarily satisfy the symmetry axiom), then the mapping d^* of $\mathcal{P}(X) \times \mathcal{P}(X)$ into $[0, \infty]$, defined by

$$d^*(A, B) = \sup_{y \in B} d(A, y)$$

for each $A, B \in \mathcal{P}(X)$ (with $d^*(A, \varnothing) = 0$), is easily verified to be a quasi-pseudometric on $\mathcal{P}(X)$. Similarly for the mapping d_*, defined by

$$d_*(A, B) = \sup_{x \in A} d(x, B)$$

for each $A, B \in \mathcal{P}(X)$ (with $d_*(\varnothing, B) = 0$). The join $d^{\sim}$ of d^* and d_* is nothing but the Hausdorff pseudometric, in the case where d is itself a pseudometric. Also, denoting by $\mathcal{U}(d)$ the quasi-uniformity induced by a quasi-pseudometric d, we have $(\mathcal{U}(d))^* = \mathcal{U}(d^*)$, $(\mathcal{U}(d))_* = \mathcal{U}(d_*)$ and $(\mathcal{U}(d))^{\sim} = \mathcal{U}(d^{\sim})$.

3. **Pervin's quasi-uniformity and hyperspaces.** In this section, we intend to show how quasi-uniformities can play a unifying role in the study of topological structures in hyperspaces.

The first step in this direction is to note that every topological space is quasi-uniformizable. W. J. Pervin has shown in [8] that every topology $\mathcal{T}$ on a set X induces a quasi-uniformity $\mathcal{U}(\mathcal{T})$ on X with the property that the topology induced by $\mathcal{U}(\mathcal{T})$ is equal to $\mathcal{T}$ itself. We shall refer to $\mathcal{U}(\mathcal{T})$ as *Pervin's quasi-uniformity* induced by $\mathcal{T}$ on X. For convenience, we recall how $\mathcal{U}(\mathcal{T})$ is constructed. If X is a set, we write

$$U_X(A) = (A \times A) \cup ((X \setminus A) \times X)$$

for each subset A of X. If $\mathcal{T}$ is a topology on X, then the sets $U_X(G)$, with G running through $\mathcal{T}$, generate $\mathcal{U}(\mathcal{T})$. If we only let G run through a subbase of $\mathcal{T}$, the sets $U_X(G)$ still generate $\mathcal{U}(\mathcal{T})$. If $\mathcal{T}'$ is also a topology on X, then it is easy to see that

$$\mathcal{U}(\mathcal{T} \vee \mathcal{T}') = \mathcal{U}(\mathcal{T}) \vee \mathcal{U}(\mathcal{T}'),$$

a useful property of Pervin's quasi-uniformity (we use the symbol "$\vee$" to denote the join of two topologies or two quasi-uniformities). In particular, it follows that $\mathcal{T}$ is finer than $\mathcal{T}'$ if and only if $\mathcal{U}(\mathcal{T})$ is finer than $\mathcal{U}(\mathcal{T}')$.

As far as hyperspaces are concerned, the most useful property of Pervin's quasi-uniformity is that it is compatible with the formation of quasi-uniformities in hyperspaces, in the following sense:

LEMMA 3.1. *Let $(X, \mathcal{T})$ be a topological space. Then we have*:
(a) $(\mathcal{U}(\mathcal{T}))^* = \mathcal{U}(\mathcal{T}^*)$.
(b) $(\mathcal{U}(\mathcal{T}))_* = \mathcal{U}(\mathcal{T}_*)$.
(c) $(\mathcal{U}(\mathcal{T}))^{\sim} = \mathcal{U}(\mathcal{T}^{\sim})$.

PROOF. N. Levine and W. J. Stager have shown that the sets $(U_X(G))^*$, with G running through $\mathcal{T}$, generate $(\mathcal{U}(\mathcal{T}))^*$, and that $(U_X(G))^* = U_{\mathcal{P}(X)}(\mathcal{P}(G))$ [5, pp. 104–105, Theorems 2.1.2 and 2.1.3]. This proves part (a). Part (b) can be proved in a similar way. Using this, we then have

$$\begin{aligned}(\mathcal{U}(\mathcal{T}))^{\sim} &= (\mathcal{U}(\mathcal{T}))^* \vee (\mathcal{U}(\mathcal{T}))_* = \mathcal{U}(\mathcal{T}^*) \vee \mathcal{U}(\mathcal{T}_*)\\ &= \mathcal{U}(\mathcal{T}^* \vee \mathcal{T}_*) = \mathcal{U}(\mathcal{T}^{\sim}),\end{aligned}$$

proving part (c).

As a direct consequence of this lemma, we have:

THEOREM 3.2. *The topologies induced by $(\mathcal{U}(\mathcal{T}))^*$, $(\mathcal{U}(\mathcal{T}))_*$ and $(\mathcal{U}(\mathcal{T}))^{\sim}$ are equal to $\mathcal{T}^*$, $\mathcal{T}_*$ and $\mathcal{T}^{\sim}$, respectively.*

We now give an example of the use of this result, and show how the following two known propositions can be deduced from a single one about quasi-uniform spaces:

I. Let $(X, \mathfrak{T})$ be a topological space. Then the topological space $(\mathfrak{F}(X), \mathfrak{T}_*)$ is a T_0-space [6, p. 163, Theorem 4.9.1].

II. Let $(X, \mathfrak{U})$ be a uniform space. Then the uniform space $(\mathfrak{F}(X), \mathfrak{U}^{\sim})$ is Hausdorff [2, p. 208, Exercise 6].

In order to carry this out, we use the fact that, in quasi-uniform spaces, the T_0 separation axiom can be characterized in terms of entourages only: a quasi-uniform space $(X, \mathfrak{U})$ with base $\mathfrak{B}$ for its set of entourages $\mathfrak{U}$ is a T_0-space if and only if the intersection of the sets of $\mathfrak{B}$ is an anti-symmetric set [7, p. 35, Theorem 3.1]. We now prove:

THEOREM 3.3. *Let $(X, \mathfrak{U})$ be a quasi-uniform space. Then the quasi-uniform space $(\mathfrak{F}(X), \mathfrak{U}_*)$ is a T_0-space.*

PROOF. We show that the intersection $\mathfrak{C}$ of the sets $U_* \cap (\mathfrak{F}(X) \times \mathfrak{F}(X))$, where U runs through $\mathfrak{U}$, is anti-symmetric. Now, because of the relation $(U_*)^{-1} = (U^{-1})^*$, the set $\mathfrak{C}^{-1}$ is nothing but the intersection of the sets $(U^{-1})^* \cap (\mathfrak{F}(X) \times \mathfrak{F}(X))$. Thus, if $(A, B) \in \mathfrak{C} \cap \mathfrak{C}^{-1}$, then

$$A \subset \bigcap_U U^{-1}(B) = B \quad \text{and} \quad B \subset \bigcap_U U^{-1}(A) = A,$$

so that $A = B$, as desired.

PROOF OF I. By Theorem 3.3, the quasi-uniform space $(\mathfrak{F}(X), (\mathfrak{U}(\mathfrak{T}))_*)$ is a T_0-space ($\mathfrak{F}(X)$ is unambiguously defined, since the topology induced by $\mathfrak{U}(\mathfrak{T})$ is equal to $\mathfrak{T}$ itself). Since the topology induced by $(\mathfrak{U}(\mathfrak{T}))_*$ is equal to $\mathfrak{T}_*$, the result follows.

PROOF OF II. Since $\mathfrak{U}_*$ is coarser than $\mathfrak{U}^{\sim}$, the conclusion of Theorem 3.3 holds with $\mathfrak{U}^{\sim}$ in place of $\mathfrak{U}_*$. Thus, the uniform space $(\mathfrak{F}(X), \mathfrak{U}^{\sim})$ is a T_0-space, hence it is Hausdorff.

Pervin's quasi-uniformity enables us to consider topological spaces as a subclass of the class of quasi-uniform spaces. In fact, it does much more than that: it transforms the morphisms of topological spaces into those of the associated quasi-uniform spaces. More precisely, we have the following result, which is Theorem 4.1 of [4, p. 101].

THEOREM 3.4. *Let $(X, \mathfrak{T})$ and $(X', \mathfrak{T}')$ be topological spaces. Then a mapping $f\colon X \to X'$ is $(\mathfrak{T}, \mathfrak{T}')$-continuous if and only if it is $(\mathfrak{U}(\mathfrak{T}), \mathfrak{U}(\mathfrak{T}'))$-q.-u. continuous.*

COROLLARY 3.5. *A sufficient condition for $\mathfrak{T}$ to be the inverse image of $\mathfrak{T}'$ under f is that $\mathfrak{U}(\mathfrak{T})$ be the inverse image of $\mathfrak{U}(\mathfrak{T}')$ under f.*

PROOF. Suppose that $\mathfrak{U}(\mathfrak{T})$ is the inverse image of $\mathfrak{U}(\mathfrak{T}')$ under f. Then f is $(\mathfrak{U}(\mathfrak{T}), \mathfrak{U}(\mathfrak{T}'))$-q.-u. continuous, hence it is $(\mathfrak{T}, \mathfrak{T}')$-continuous. Now suppose $\mathfrak{T}''$ is a topology on X which is such that f is $(\mathfrak{T}'', \mathfrak{T}')$-continuous. Then f is $(\mathfrak{U}(\mathfrak{T}''), \mathfrak{U}(\mathfrak{T}'))$-q.-u. continuous, and so $\mathfrak{U}(\mathfrak{T}'')$ is finer than $\mathfrak{U}(\mathfrak{T})$. It follows that $\mathfrak{T}''$ is finer than $\mathfrak{T}$. Thus, $\mathfrak{T}$ is the inverse image of $\mathfrak{T}'$ under f.

In fact, the condition stated in the above result is also necessary (cf. [4, p. 102, Theorem 4.6]).

As an example of the usefulness of Corollary 3.5, we consider the mapping $A \mapsto \bar{A}$ of $\mathcal{P}(X)$ into itself. More generally:

THEOREM 3.6. *Let $(X, \mathcal{U})$ be a quasi-uniform space, and let f: $\mathcal{P}(X) \to \mathcal{P}(X)$ be any mapping such that $A \subset f(A) \subset \bar{A}$ for each $A \in \mathcal{P}(X)$. Then the quasi-uniformity $\mathcal{U}_*$ of $\mathcal{P}(X)$ is the inverse image of itself under the mapping f.*

PROOF. The result will follow if we can show that

$$U_* \subset (f \times f)^{-1}\big((U \circ U)_*\big) \quad \text{and} \quad (f \times f)^{-1}\big(U_*\big) \subset (U \circ U)_*$$

for each entourage U in $\mathcal{U}$, i.e. that the implications

$$A \subset U^{-1}(B) \Rightarrow f(A) \subset (U \circ U)^{-1}(f(B))$$

and

$$f(A) \subset U^{-1}(f(B)) \Rightarrow A \subset (U \circ U)^{-1}(B)$$

hold. Now, if $A \subset U^{-1}(B)$, then

$$f(A) \subset \bar{A} \subset U^{-1}(A) \subset U^{-1}\big(U^{-1}(B)\big) \subset U^{-1}\big(U^{-1}(f(B))\big),$$

while if $f(A) \subset U^{-1}(f(B))$, then

$$A \subset f(A) \subset U^{-1}(f(B)) \subset U^{-1}(\bar{B}) \subset U^{-1}\big(U^{-1}(B)\big).$$

Hence the result.

With f defined as in Theorem 3.6, we can now deduce:

COROLLARY 3.7. *Let $(X, \mathcal{T})$ be a topological space. Then the topology $\mathcal{T}_*$ of $\mathcal{P}(X)$ is the inverse image of itself under the mapping f.*

PROOF. By Theorem 3.6, the quasi-uniformity $(\mathcal{U}(\mathcal{T}))_*$ is the inverse image of itself under f. Since $(\mathcal{U}(\mathcal{T}))_* = \mathcal{U}(\mathcal{T}_*)$, the result follows by using Corollary 3.5.

COROLLARY 3.8. *Let $(X, \mathcal{U})$ be a uniform space. Then each of the quasi-uniformities $\mathcal{U}^*$, $\mathcal{U}_*$ and $\mathcal{U}^\sim$ of $\mathcal{P}(X)$ is the inverse image of itself under the mapping f.*

PROOF. Since the result holds for $\mathcal{U}_*$ by Theorem 3.6, and since $\mathcal{U}^*$ and $\mathcal{U}_*$ are conjugate, the result also holds for $\mathcal{U}^*$. It then follows that the result holds for the join $\mathcal{U}^\sim$ of $\mathcal{U}^*$ and $\mathcal{U}_*$.

4. **Upper semicontinuity of infinite products.** As a final example of the use of quasi-uniformities in hyperspaces, we consider products. Let us first make some notational remarks. If $(X_i, \mathcal{U}_i)_{i \in I}$ is a family of quasi-uniform spaces, we denote the product quasi-uniformity on $\times_{i \in I} X_i$ by $\times_{i \in I} \mathcal{U}_i$. Adopting a similar notation for product topologies, we have

$$\mathcal{T}\Big(\underset{i \in I}{\times} \mathcal{U}_i\Big) = \underset{i \in I}{\times} \mathcal{T}(\mathcal{U}_i).$$

THEOREM 4.1. *Let $(X_i, \mathcal{U}_i)_{i\in I}$ be a family of quasi-uniform spaces, and let F be the mapping $(A_i)_{i\in I} \mapsto \times_{i\in I} A_i$ of $\times_{i\in I}\mathcal{P}(X_i)$ into $\mathcal{P}(\times_{i\in I} X_i)$. Then F induces a quasi-uniform isomorphism of $(\times_{i\in I}\mathcal{P}_0(X_i), \times_{i\in I}\mathcal{U}_i^*)$ onto a subspace of $(\mathcal{P}_0(\times_{i\in I}X_i), (\times_{i\in I}\mathcal{U}_i)^*)$.*

PROOF. First of all, it is clear that F induces a bijection of $\times_{i\in I}\mathcal{P}_0(X_i)$ onto a subset of $\mathcal{P}_0(\times_{i\in I}X_i)$. Now put $G = F \times F$. For each $j \in I$, let

$$p_j\colon \underset{i\in I}{\times} X_i \to X_j \quad \text{and} \quad P_j\colon \underset{i\in I}{\times} \mathcal{P}(X_i) \to \mathcal{P}(X_j)$$

be the projections, and put $g_j = p_j \times p_j$ and $G_j = P_j \times P_j$.

For each $i \in I$, the sets U_i^* form a base for $\mathcal{U}_i^*$, as U_i runs through $\mathcal{U}_i$. Thus, if $\mathcal{C}$ is the set of all sets of the form $G_i^{-1}(U_i^*)$, where $i \in I$ and $U_i \in \mathcal{U}_i$, then the set of all finite intersections

$$W^*(U_{i_1}, \ldots, U_{i_n}) = G_{i_1}^{-1}(U_{i_1}^*) \cap \cdots \cap G_{i_n}^{-1}(U_{i_n}^*)$$

of sets of $\mathcal{C}$ is a base of $\times_{i\in I}\mathcal{U}_i^*$.

On the other hand, let $\mathcal{B}$ be the set of all sets of the form $g_i^{-1}(U_i)$, where $i \in I$ and $U_i \in \mathcal{U}_i$. Then the set of all finite intersections

$$W(U_{i_1}, \ldots, U_{i_n}) = g_{i_1}^{-1}(U_{i_1}) \cap \cdots \cap g_{i_n}^{-1}(U_{i_n})$$

of sets of $\mathcal{B}$ is a base for $\times_{i\in I}\mathcal{U}_i$. It follows that the sets $(W(U_{i_1}, \ldots, U_{i_n}))^*$ form a base for $(\times_{i\in I}\mathcal{U}_i)^*$.

Putting $Y = \times_{i\in I}\mathcal{P}_0(X_i)$, the result will thus follow if we can show that

$$G^{-1}\big((W(U_{i_1}, \ldots, U_{i_n}))^*\big) \cap (Y \times Y) = W^*(U_{i_1}, \ldots, U_{i_n}) \cap (Y \times Y).$$

Now, for each $((A_i)_{i\in I}, (B_i)_{i\in I}) \in Y \times Y$, we have

$$\begin{aligned} ((A_i)_{i\in I}, (B_i)_{i\in I}) &\in G^{-1}\big((W(U_{i_1}, \ldots, U_{i_n}))^*\big) \\ &\Leftrightarrow \Big(\underset{i\in I}{\times} A_i, \underset{i\in I}{\times} B_i\Big) \in (W(U_{i_1}, \ldots, U_{i_n}))^* \\ &\Leftrightarrow \underset{i\in I}{\times} B_i \subset W(U_{i_1}, \ldots, U_{i_n})\Big(\underset{i\in I}{\times} A_i\Big) \\ &\Leftrightarrow \forall (y_i)_{i\in I} \in \underset{i\in I}{\times} B_i \;\exists (x_i)_{i\in I} \in \underset{i\in I}{\times} A_i \end{aligned}$$

such that $(x_{i_k}, y_{i_k}) \in U_{i_k}$ $\forall k = 1, \ldots, n$, whereas

$$\begin{aligned} ((A_i)_{i\in I}, (B_i)_{i\in I}) &\in W^*(U_{i_1}, \ldots, U_{i_n}) \\ &\Leftrightarrow (A_{i_k}, B_{i_k}) \in U_{i_k}^* \;\forall k = 1, \ldots, n \\ &\Leftrightarrow B_{i_k} \subset U_{i_k}(A_{i_k}) \;\forall k = 1, \ldots, n. \end{aligned}$$

Since the sets A_i and B_i are all nonempty, the result follows.

It is clear that this theorem remains true if upper quasi-uniformities are replaced everywhere by the corresponding lower or Bourbaki quasi-uniformities.

We now turn to topological spaces. Part (a) of the following theorem is a proposition of structural character which we state for its own sake. Part (b) is the formulation we need to obtain the corollary that follows.

THEOREM 4.2. *Let $(X_i, \mathfrak{T}_i)_{i \in I}$ be a family of topological spaces, and let F be the mapping*

$$(A_i)_{i \in I} \mapsto \mathop{\times}_{i \in I} A_i$$

of $(\times_{i \in I} \mathcal{P}_0(X_i), \times_{i \in I} \mathfrak{T}_i^*)$ *into* $(\mathcal{P}_0(\times_{i \in I} X_i), (\times_{i \in I} \mathfrak{T}_i)^*)$.

(a) *F induces a homeomorphism of $\times_{i \in I} \mathcal{K}_0(X_i)$ onto a subspace of $\mathcal{K}_0(\times_{i \in I} X_i)$.*

(b) *F is continuous at every $(A_i)_{i \in I} \in \times_{i \in I} \mathcal{K}_0(X_i)$.*

PROOF. By Theorem 4.1, the mapping F induces a quasi-uniform isomorphism, and hence a homeomorphism, of the space

$$\left(\times_{i \in I} \mathcal{P}_0(X_i), \times_{i \in I} \left(\mathfrak{U}(\mathfrak{T}_i)\right)^*\right)$$

onto a subspace of $(\mathcal{P}_0(\times_{i \in I} X_i), (\times_{i \in I} \mathfrak{U}(\mathfrak{T}_i))^*)$. Now

$$\mathfrak{T}\left(\mathop{\times}_{i \in I} (\mathfrak{U}(\mathfrak{T}_i))^*\right) = \mathop{\times}_{i \in I} \mathfrak{T}((\mathfrak{U}(\mathfrak{T}_i))^*) = \mathop{\times}_{i \in I} \mathfrak{T}_i^*$$

by Theorem 3.2, while, by Theorem 2.5,

$$\mathfrak{T}\left(\left(\mathop{\times}_{i \in I} \mathfrak{U}(\mathfrak{T}_i)\right)^*\right) = \left(\mathfrak{T}\left(\mathop{\times}_{i \in I} \mathfrak{U}(\mathfrak{T}_i)\right)\right)^* = \left(\mathop{\times}_{i \in I} \mathfrak{T}_i\right)^*$$

on the set $\mathcal{K}_0(\times_{i \in I} X_i)$, so that part (a) is proved.

To prove (b), note that, if $\times_{i \in I} A_i$ is compact, then every $(\times_{i \in I} \mathfrak{T}_i)^*$-neighbourhood of $\times_{i \in I} A_i$ in $\mathcal{P}_0(\times_{i \in I} X_i)$ contains a $\mathfrak{T}((\times_{i \in I} \mathfrak{U}(\mathfrak{T}_i))^*)$-neighbourhood of $\times_{i \in I} A_i$, by Lemma 2.4.

COROLLARY 4.3. *Let $(X, \mathfrak{T})$ be a topological space, and let $(X_i, \mathfrak{T}_i)_{i \in I}$ be a family of topological spaces. For each $i \in I$, let Γ_i be a multi-valued mapping of $(X, \mathfrak{T})$ into $(X_i, \mathfrak{T}_i)$ such that $\Gamma_i(x) \neq \varnothing$ for each $x \in X$. Let Γ be the Cartesian product of the family $(\Gamma_i)_{i \in I}$, and let $x_0 \in X$. Suppose that, for each $i \in I$, Γ_i is u.s.c. at x_0 and $\Gamma_i(x_0)$ is compact. Then Γ is u.s.c. at x_0.*

This is an advance over previous results, where the proof rests heavily on the finiteness of the index set I (cf. [1, p. 114, Theorem 4′]). Also note that Corollary 4.3 remains true without the assumption that the sets $\Gamma_i(x)$ be nonempty, as one easily verifies.

REFERENCES

1. C. Berge, *Topological spaces*, Oliver and Boyd, Edinburgh, 1963.

2. N. Bourbaki, *Elements of mathematics. General topology*. Part I, Addison-Wesley, Reading, Mass., 1966.

3. K. Kuratowski, *Topology*. Vols. I,II, Academic Press, New York; Polish Scientific Publishers, Warsaw, 1966.

4. N. Levine, *On Pervin's quasi uniformity*, Math. J. Okayama Univ. **14** (1970), 97–102.

5. N. Levine and W. J. Stager, *On the hyperspace of a quasi-uniform space*, Math. J. Okayama Univ. **15** (1971–72), 101–106.

6. E. Michael, *Topologies on spaces of subsets*, Trans. Amer. Math. Soc. **71** (1951), 152–182.

7. M. G. Murdeshwar and S. A. Naimpally, *Quasi-uniform topological spaces*, Noordhoff, Groningen, 1966.

8. W. J. Pervin, *Quasi-uniformization of topological spaces*, Math. Ann. **147** (1962), 316–317.

INSTITUT FÜR MATHEMATIK, TECHNISCHE UNIVERSITÄT MÜNCHEN, POSTFACH 20 24 20, 8000 MÜNCHEN 2, FEDERAL REPUBLIC OF GERMANY