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MINIMAL INJECTIVE RESOLUTIONS UNDER FLAT BASE CHANGE

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ABSTRACT. For a flat morphism $\varphi: A \to B$ of noetherian rings, the minimal injective resolution of the B-module $M \otimes_A B$ is described in terms of the minimal injective resolution of the finitely generated A-module M and the minimal injective resolutions of the fibers of φ.

Throughout this note A and B denote commutative noetherian rings with multiplicative identities. For an A-module M, a prime ideal $\mathfrak{p}$ in A, and an integer i the (cardinal) number $\mu_A^i(\mathfrak{p}, M)$ denotes the dimension of $\operatorname{Ext}_A^i(A/\mathfrak{p}, M)_\mathfrak{p} = \operatorname{Ext}_{A_\mathfrak{p}}^i(k(\mathfrak{p}), M_\mathfrak{p})$ considered as a vector-space over the field $k(\mathfrak{p}) = A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$. Now let

$$0 \to M \to I^0 \to I^1 \to \cdots \to I^i \to \ldots$$

be a minimal injective resolution of M. Then $\mu_A^i(\mathfrak{p}, M) = \mu_A^0(\mathfrak{p}, I^i)$ and this is the number of copies of the injective hull $E_A(A/\mathfrak{p})$ of $A/\mathfrak{p}$ in the decomposition of I^i into indecomposable injective modules. For this fact, and for other facts concerning the numbers $\mu_A^i(\mathfrak{p}, M)$ and minimal injective resolutions, consult Bass' paper [1] (in particular, §2).

Now let $\varphi: A \to B$ be a ringhomomorphism making B into a flat A-module, and let M be a finitely generated A-module. The main result of this note describes the minimal injective resolution of the B-module $M \otimes_A B$ by expressing the number $\mu_B^n(\mathfrak{q}, M \otimes_A B)$ (for each n and each prime ideal $\mathfrak{q}$ in B) in terms of the numbers $\mu_A^q(\mathfrak{q} \cap A, M)$ and $\mu_C^p(\mathfrak{q}C, C)$ where C is the fiber of φ at $\mathfrak{q} \cap A$. All the numbers here are finite, since M is finitely generated.

THEOREM. *Let $\varphi: A \to B$ be a flat ringhomomorphism, let $\mathfrak{q}$ be a prime ideal in B, and let C denote the ring $B/(\mathfrak{q} \cap A)B$ with the prime ideal $\mathfrak{q}' = \mathfrak{q}C$. Then for a finitely generated A-module M and for all numbers n there is an equality of numbers*

$$\mu_B^n(\mathfrak{q}, M \otimes_A B) = \sum_{p+q=n} \mu_C^p(\mathfrak{q}', C)\, \mu_A^q(\mathfrak{q} \cap A, M).$$

This result will be proved below, but first we mention two immediate

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applications to the injective dimensions of the A-module M and the B-module $M \otimes_A B$. These dimensions are here denoted by $\mathrm{id}_A M$ and $\mathrm{id}_B(M \otimes_A B)$ respectively (and they might be infinite).

COROLLARY 1. *Let A and B be local rings and let $\varphi: A \to B$ be a flat local ringhomomorphism. If M is a finitely generated nonzero A-module, then*

$$\mathrm{id}_B(M \otimes_A B) = \mathrm{id}_A M + \mathrm{id}_C C,$$

where $C = B/\mathfrak{m}B$ and $\mathfrak{m}$ is the maximal ideal in A.

In particular, $\mathrm{id}_B(M \otimes_A B)$ *is finite if and only if* $\mathrm{id}_A M$ *is finite and C is a Gorenstein ring.*

PROOF. Corollary 1 follows directly from the Theorem since $\mathrm{id}_A M = \sup\{i \mid \mu_A^i(\mathfrak{m}, M) \neq 0\}$ (and similarly for $\mathrm{id}_B(M \otimes_A B)$ and $\mathrm{id}_C C$) (cf. [1, (3.2) Corollary]). $\square$

REMARK 1. The main result of [4] is a result in the same direction as Corollary 1. It states that if $\varphi: A \to B$ is a flat ringhomomorphism and if E is an injective A-module, then

$$\mathrm{id}_B(E \otimes_A B) = \sup\{\mathrm{id}_{F(\mathfrak{p})} F(\mathfrak{p}) \mid \mathfrak{p} \in \mathrm{Ass}_A E\}$$

where $F(\mathfrak{p}) = k(\mathfrak{p}) \otimes_A B$ is the fiber at $\mathfrak{p}$.

Suppose that A is local with maximal ideal $\mathfrak{m}$ and let $\hat{A}$ denote the completion of A (in the $\mathfrak{m}$-adic topology). If there exists a Gorenstein module over A (that is, a finitely generated module G with depth $G = \mathrm{id}\, G < \infty$), then all the fibers of $A \to \hat{A}$ are Gorenstein rings (cf. [9, (2.8) Theorem] and consult [8] and [3] for facts about Gorenstein modules). The next result generalizes this to arbitrary finitely generated modules of finite injective dimension.

COROLLARY 2. *Let M be a finitely generated nonzero module over the local ring A. If $\mathrm{id}_A M < \infty$, then the formal fiber $C = \hat{A} \otimes_A k(\mathfrak{p})$ at $\mathfrak{p}$ is a Gorenstein ring for all $\mathfrak{p}$ in the support of M.*

PROOF (OF COROLLARY 2). The local rings of C are of the form $C_{\mathfrak{q}C}$ where $\mathfrak{q}$ is a prime ideal in $\hat{A}$ lying over $\mathfrak{p}$. Now apply Corollary 1 to the homomorphism $A_\mathfrak{p} \to \hat{A}_\mathfrak{q}$ using the fact that the $\hat{A}_\mathfrak{q}$-module $M_\mathfrak{p} \otimes_{A_\mathfrak{p}} \hat{A}_\mathfrak{q} = \hat{M}_\mathfrak{q}$ is of finite injective dimension. $\square$

EXAMPLE. Ferrand and Raynaud have constructed a domain of dimension one such that the generic formal fiber $\hat{A} \otimes_A Q$ is not a Gorenstein ring (Q is the field of fractions), cf. [2]. In particular, this ring A has no Gorenstein modules, and Corollary 2 shows even more: If the finitely generated module M over this ring A is of finite injective dimension, then M is artinian.

A third application of the Theorem can be found in [5]. Namely: If M is a finitely generated A-module and $\mathfrak{p} \subseteq \mathfrak{q}$ are two prime ideals, then $\mu_A^i(\mathfrak{p}, M) \leqslant \mu_A^{i+l}(\mathfrak{q}, M)$ for all i, where $l = \dim(A_\mathfrak{q}/\mathfrak{p}A_\mathfrak{q})$.

PROOF OF THE THEOREM. Since the μ^i is unchanged under localization we may place ourself in the following situation: $\mathfrak{p}$ and $\mathfrak{q}$ are maximal ideals in, respectively, A and B such that $\varphi(\mathfrak{p}) \subseteq \mathfrak{q}$. We are then required to prove

$$\mu_B^n(\mathfrak{q}, M \otimes_A B) = \sum_{p+q=n} \mu_C^p(\mathfrak{q}C, C)\,\mu_A^q(\mathfrak{p}, M) \tag{0}$$

for all n, when $C = B/\mathfrak{p}B$.

To prove this we shall for any A-module M construct C-linear isomorphisms

$$\mathrm{Ext}_B^n(l, M \otimes_A B) \simeq \coprod_{p+q=n} \mathrm{Ext}_C^p(l, C) \otimes_k \mathrm{Ext}_A^q(k, M) \tag{1}$$

where $k = A/\mathfrak{p}$ and $l = B/\mathfrak{q} = C/\mathfrak{q}C$. Then for M finitely generated we get the desired formula by counting dimensions over l.

During the proof we work in the category of complexes of modules over the different rings considered. Recall that a morphism $\alpha\colon X \to Y$ between complexes of modules over a ring R is called a *quasi-isomorphism* if it induces isomorphisms $H^i(\alpha)\colon H^i(X) \xrightarrow{\sim} H^i(Y)$ on the cohomology modules for all i. This property of a morphism is preserved by any of the functors

$$\mathrm{Hom}_R(P, -), \quad \mathrm{Hom}_R(-, I), \quad \text{and} \quad - \otimes_R M,$$

where P is a bounded above complex of R-projective modules, I is a bounded below complex of R-injective modules and M is an R-flat module, cf. [6, Chapter I, Lemma 6.2, p. 64].

To construct the isomorphisms in (1) we choose a minimal A-injective resolution I of M and a quasi-isomorphism $I \otimes_A B \to J$ where J is a complex of B-injective modules with $J^i = 0$ for $i < 0$, cf. [6, Chapter I, Lemma 4.6, p. 42]. Choose, furthermore, a C-projective resolution P of l. Then we have quasi-isomorphisms

$$M \to I, \quad I \otimes_A B \to J, \quad \text{and} \quad P \to l.$$

The isomorphisms in (1) are now established by defining quasi-isomorphisms α^* and β_*

$$\begin{array}{ccc} \mathrm{Hom}_B(l, J) & & \mathrm{Hom}_C(P, \mathrm{Hom}_A(k, I) \otimes_k C) \\ \Big\downarrow{\scriptstyle \alpha^*} & & \Big\downarrow{\scriptstyle \beta_*} \\ \mathrm{Hom}_B(P, J) & \xrightarrow{\ \sim\ } & \mathrm{Hom}_C(P, \mathrm{Hom}_B(C, J)) \end{array} \tag{2}$$

and identifying the two sides of (1) with the cohomology of the two sides of (2). Here α^* is induced by the quasi-isomorphism $\alpha\colon P \to l$ and since J is a complex of B-injective modules we see that α^* is a quasi-isomorphism. And β_* is induced by the composite C-linear morphism β defined by the commutative diagram

$$\begin{array}{ccccc} \mathrm{Hom}_A(k, I) \otimes_k C & & \xrightarrow{\quad\quad\beta\quad\quad} & & \mathrm{Hom}_B(C, J) \\ \Big| \wr & & & & \Big\| \\ \mathrm{Hom}_A(k, I) \otimes_A B & \longrightarrow & \mathrm{Hom}_B(k \otimes_A B, I \otimes_A B) & \longrightarrow & \mathrm{Hom}_B(k \otimes_A B, J). \end{array}$$

To prove that β is a quasi-isomorphism we choose a resolution F of k by finitely generated A-free modules. Then we have a quasi-isomorphism $F \to k$ and a commutative diagram of complexes of B-modules

$$\begin{tikzcd}
\mathrm{Hom}_A(k, I) \otimes_A B \arrow[r] \arrow[d, "\gamma_1"'] & \mathrm{Hom}_B(k \otimes_A B, I \otimes_A B) \arrow[r] \arrow[d] & \mathrm{Hom}_B(k \otimes_A B, J) \arrow[d, "\gamma_2"] \\
\mathrm{Hom}_A(F, I) \otimes_A B \arrow[r, "\kappa"'] & \mathrm{Hom}_B(F \otimes_A B, I \otimes_A B) \arrow[r, "\delta"'] & \mathrm{Hom}_B(F \otimes_A B, J).
\end{tikzcd}$$

The morphism γ_1 is a quasi-isomorphism since I consists of A-injective modules and B is A-flat. The morphism κ is an isomorphism since each F^i is finitely generated A-free. The morphism δ is a quasi-isomorphism since $F \otimes_A B$ consists of B-free modules and $I \otimes_A B \to J$ is a quasi-isomorphism. The morphism γ_2 is a quasi-isomorphism since J consists of B-injective modules and $F \otimes_A B \to k \otimes_A B$ is a quasi-isomorphism by flatness of B. As β is a quasi-isomorphism and P a complex of C-projective modules it follows that β_* is a quasi-isomorphism.

Passing to cohomology we see by flatness of B that $M \otimes_A B \to I \otimes_A B$ is a quasi-isomorphism and, hence, the composite $M \otimes_A B \to I \otimes_A B \to J$ is a quasi-isomorphism, i.e. J is a B-injective resolution of $M \otimes_A B$. Consequently,

$$H^n[\mathrm{Hom}_B(l, J)] = \mathrm{Ext}_B^n(l, M \otimes_A B).$$

To compute the cohomology of $\mathrm{Hom}_C(P, \mathrm{Hom}_A(k, I) \otimes_k C)$, we remark, that since I is a minimal resolution, the differentials in $\mathrm{Hom}_A(k, I)$ are zero. Then this holds for $\mathrm{Hom}_A(k, I) \otimes_k C$ as well and so the differentials in $\mathrm{Hom}_C(P, \mathrm{Hom}_A(k, I) \otimes_k C)$ are induced from P only. And we get for the cohomology

$$\begin{aligned}
&H^n[\mathrm{Hom}_C(P, \mathrm{Hom}_A(k, I) \otimes_k C)] \\
&\quad = \coprod_{p+q=n} H^p[\mathrm{Hom}_C(P, \mathrm{Hom}_A(k, I^q) \otimes_k C)] \\
&\quad = \coprod_{p+1=n} \mathrm{Ext}_C^p(l, \mathrm{Hom}_A(k, I^q) \otimes_k C) \\
&\quad = \coprod_{p=q=n} \mathrm{Ext}_C^p(l, C) \otimes_k \mathrm{Hom}_A(k, I^q) \\
&\quad = \coprod_{p+q=n} \mathrm{Ext}_C^p(l, C) \otimes_k \mathrm{Ext}_A^q(k, M). \quad \square
\end{aligned}$$

REMARK 2. The proof of the quasi-isomorphisms in (2) works for any base change situation

$$\begin{tikzcd}
B \arrow[r] & C = B \otimes_A k \\
A \arrow[r] \arrow[u] & k \arrow[u]
\end{tikzcd}$$

where $A \to B$ is a flat homomorphism of noetherian rings and $A \to k$ is a (module-) finite ringhomomorphism and l is any finitely generated C-module. In the derived category $D(C)$ (2) becomes an isomorphism

$$(2') \qquad \mathbf{R}\mathrm{Hom}_B(l, M \otimes_A B) \simeq \mathbf{R}\mathrm{Hom}_C(l, \mathbf{R}\mathrm{Hom}_A(k, M) \otimes_k C)$$

(in the notation of [6]). However, the calculation of the cohomology of the right-hand side of (2′) made in the end of the proof requires some extra conditions on the ring k, such as being a field.

REMARK 3. Corresponding to (2) we have also a spectral sequence:

$$E_2^{pq} = \mathrm{Ext}_C^p(l, \mathrm{Ext}_A^q(k, M) \otimes_k C) \Rightarrow \mathrm{Ext}_B^n(l, M \otimes_A B).$$

In the situation described at the beginning of the proof of the Theorem this spectral sequence gives the inequality

$$\mu_B^n(\mathfrak{q}, M \otimes_A B) \leqslant \sum_{p+q=n} \mu_C^p(\mathfrak{q}C, C)\,\mu_A^q(\mathfrak{p}, M).$$

This inequality has been studied in some special cases by Paugam in [7]. The equality (0) shows that all the differentials $E_r^{pq} \to E_r^{p+r,q-r+1}$ are zero (for $r \geqslant 2$).

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