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THE BANACH-STONE PROPERTY AND THE WEAK BANACH-STONE PROPERTY IN THREE-DIMENSIONAL SPACES

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ABSTRACT. Let X and Y be compact Hausdorff spaces, E a Banach space, and $C(X, E)$ the space of continuous functions on X to E. E has the weak Banach-Stone property if, whenever $C(X, E)$ and $C(Y, E)$ are isometric, then X and Y are homeomorphic. E has the Banach-Stone property if the descriptive as well as the topological conclusions of the Banach-Stone theorem for scalar functions remain valid in the case of isometries of $C(X, E)$ onto $C(Y, E)$. These two properties were first studied by M. Jerison, and it we later shown that every space E found by Jerison to have the weak Banach-Stone property actually has the Banach-Stone property, thus raising the question of whether the two properties are distinct. Here we characterize all three-dimensional spaces with the weak Banach-Stone property, and, in so doing, show the properties to be distinct.

Throughout this article X and Y will denote compact Hausdorff spaces, E a Banach space, and $C(X, E)$ the space of continuous functions on X to E. $\mathfrak{B}(E)$ will denote the space of bounded operators on E, given its strong operator topology, and $C(X)$ the space of continuous functions on X to the scalar field associated with E.

We will say that E has the *Banach-Stone property* if, given any isometry A of $C(X, E)$ onto $C(Y, E)$, there exists a homeomorphism τ of Y onto X and a continuous function $y \to \mathfrak{A}_y$ from Y into $\mathfrak{B}(E)$ such that, for all $y \in Y$, $\mathfrak{A}_y$ is an isometry of E onto itself, and such that $(A(F))(y) = \mathfrak{A}_y F(\tau(y))$ for $F \in C(X, E)$, $y \in Y$–i.e. if the Banach-Stone theorem for $C(X)$ can be completely generalized for $C(X, E)$. E has the *weak Banach-Stone property* if the existence of an isometry A of $C(X, E)$ onto $C(Y, E)$ implies that X and Y are homeomorphic.

The Banach-Stone and weak Banach-Stone properties were first studied in [4] by M. Jerison, who showed that every E belonging to the family of strictly convex spaces has the former property, and that all spaces E belonging to a larger family have the latter. The weak Banach-Stone property has also been investigated by K. Sundaresan, who showed in [6] that for every positive integer $n \geqslant 2$, the space l_n^∞ fails to have the weak Banach-Stone property.

In [1] a complete characterization of all finite-dimensional Banach spaces

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with the Banach-Stone property was obtained, and in [2] that same characterization was shown to hold within the family of reflexive Banach spaces. We say that a Banach space E *splits* if E can be written as the direct sum of two nonzero subspaces U and V, $E = U \oplus V$, and has norm given by $\|e\| = \max\{\|u\|, \|v\|\}$ for $e = u + v \in E$. The characterization obtained in [1] and [2] is that if E is a reflexive Banach space, then E has the Banach-Stone property if, and only if, E does not split [1, Theorems 1 and 2] and [2, Theorem].

It has also been established, [1, p. 92 and Theorem 2, p. 97], that every Banach space E shown by Jerison in [4] to have the weak Banach-Stone property actually has the Banach-Stone property. Thus the question arises whether or not the two properties are indeed distinct. If E is two dimensional, then it is trivially true that E splits if, and only if, E is isometric to l_2^∞, so that by the results of [1] and [6] the properties do coincide for two-dimensional Banach spaces E.

In this article we characterize all three-dimensional spaces with the weak Banach-Stone property. It is shown that the only real (resp. complex) three-dimensional Banach space which fails to have the weak Banach-Stone property is the real (resp. complex) space l_3^∞. Now it is easy to find three-dimensional spaces which split, and yet are not isometric to l_3^∞. (For example, let U be a two-dimensional Hilbert space, let V be the corresponding scalar field, and form $E = U \oplus V$ giving E the max norm.) Since such spaces have the weak Banach-Stone property, but not the Banach-Stone property, it is a consequence of the result obtained here that the two properties are distinct.

In this article we make use of the concept of a T-set as introduced by S. B. Myers in [5]. If E is any Banach space, a subset T of E is called a T-set if, whenever $\{e_1, \dots, e_n\}$ is any finite subset of T, then $\|\sum_{j=1}^n e_j\| = \sum_{j=1}^n \|e_j\|$, and T is maximal with respect to this property. We also use I. Singer's characterization of $C(X, E)^*$ as the Banach space of all regular Borel vector measures $\mathbf{m}$ on X to E^*, with finite variation $|\mathbf{m}|$, and norm given by $\|\mathbf{m}\| = |\mathbf{m}|(X)$, see [3, p. 387]. For $x \in X$, μ_x will denote the scalar measure which is the positive unit mass concentrated at x, and we note for future reference that if $\varphi \in E^*$, then $\varphi \cdot \mu_x \in C(X, E)^*$.

Elements of E will be denoted by e, u and v and those of E^* by φ and ψ. The value of φ at e is denoted by $\langle e, \varphi \rangle$. If $E = U \oplus V$, and if we write an element $e \in E$ as $e = u + v$, it is always implicit that $u \in U$ and $v \in V$. We denote elements of $C(X, E)$ and those of $C(Y, E)$, respectively, by F and G, while elements of $C(X)$ and $C(Y)$ are denoted, respectively, by f and g. The norms in E and E^* will be denoted by $\|\cdot\|$, while norms in $C(X, E)$, $C(Y, E)$, $C(X)$ and $C(Y)$ are denoted by $\|\cdot\|_\infty$.

If $E \neq \{0\}$ is a Banach space, and T is a T-set in E, then there exists an element $\varphi \in E^*$ with $\|\varphi\| = 1$ such that $\langle e, \varphi \rangle = \|e\|$ for $e \in T$, [1, Proposition 1]. Let $\mathfrak{F} = \{\varphi \in E^*\colon \|\varphi\| = 1 \text{ and } \langle e, \varphi \rangle = \|e\| \text{ for all } e$ belonging to some T-set $T \subseteq E\}$. If E is finite dimensional, then by a *T-basis* for E^* we mean a basis consisting of elements of $\mathfrak{F}$. Such bases always exist, [1, Proposition 2]. The following lemma is easily established.

LEMMA 1. *Let E be a Banach space that splits, $E = U \oplus V$. Then U^* is isometrically isomorphic to V^0, the annihilator of V in E^*, under the map which sends an element $\varphi \in U^*$ to that element $\varphi' \in V^0 \subseteq E^*$ defined by $\langle e, \varphi' \rangle = \langle u, \varphi \rangle$, for $e = u + v \in E$.*

Henceforth, we will cease to distinguish, in notation, between an element of U^* and its image under the isometry of the lemma. The same symbol φ may denote both an element of U^* and its image in V^0 under the above correspondence.

LEMMA 2. *Let E be a Banach space that splits, $E = U \oplus V$. Then given a T-set T in U, the set $\mathbf{T} = \{u + v \in E\colon u \in T$ and v is an element of V with $\|v\| \leqslant \|u\|\}$ is a T-set in E.*

PROOF. One readily verifies that norm is an additive function on finite subsets of $\mathbf{T}$. We show that $\mathbf{T}$ is maximal in E with respect to this property.

If $\mathbf{T}$ were not maximal, there would exist an element $e_0 = u_0 + v_0 \in E - \mathbf{T}$ such that $\|e_0 + e\| = \|e_0\| + \|e\|$ for all $e \in \mathbf{T}$. Since $e_0 \notin \mathbf{T}$, either (i) $\|v_0\| > \|u_0\|$, or (ii) $\|v_0\| \leqslant \|u_0\|$ but $u_0 \notin T$.

If (i) holds, choose $u \in T$ such that $\|u\| = \|v_0\| - \|u_0\|$. (This is possible, since T is a cone, [5, Lemma 2.1, p. 133].) Then $u \in \mathbf{T}$ and $\|e_0 + u\| = \|v_0\| < \|v_0\| + \|u\| = \|e_0\| + \|u\|$, contradicting the fact that e_0 must add in norm with every element of $\mathbf{T}$.

If (ii) holds, again by [5, Lemma 2.1] there exists an element $u \in T$ such that $\|u_0 + u\| < \|u_0\| + \|u\|$. Then $u \in \mathbf{T}$. We have

$$\|e_0\| + \|u\| = \|u_0\| + \|u\| > \|u_0 + u\|$$

and

$$\|e_0\| + \|u\| = \|u_0\| + \|u\| \geqslant \|v_0\| + \|u\| > \|v_0\|,$$

so that $\|e_0 + u\| = \max\{\|u_0 + u\|, \|v_0\|\} < \|e_0\| + \|u\|$, and we again reach a contradiction. The proof of the lemma is thus complete.

As an immediate consequence of Lemmas 1 and 2 we have the following:

LEMMA 3. *If $\varphi \in U^*$, $\|\varphi\| = 1$, and $\langle u, \varphi \rangle = \|u\|$ for all u belonging to some T-set $T \subseteq U$, then considered as an element of $V^0 \subseteq E^*$, $\langle e, \varphi \rangle = \|e\|$ for all e belonging to some T-set $\mathbf{T} \subseteq E$.*

LEMMA 4. *Let E be a three-dimensional Banach space that splits, $E = U \oplus V$, where U is two dimensional. Suppose that U does not split, and let $\{\varphi_1, \varphi_2\}$ be a T-basis for $U^* = V^0 \subseteq E^*$. If A is an isometry of $C(X, E)$ onto $C(Y, E)$ then for each $x \in X$ we have*

$$A^{*-1}(\varphi_i \cdot \mu_x) = \varphi_i' \cdot \mu_y, \qquad i = 1, 2,$$

where the φ_i' *are elements of* E^* *with* $\|\varphi_i'\| = 1$, *and* y *is an element of* Y *which depends only on* x.

PROOF. By Lemma 3 and [1, Lemma 2, p. 94], for each $x \in X$ and $i = 1, 2$, $A^{*-1}(\varphi_i \cdot \mu_x)$ is of the form $\varphi_i' \cdot \mu_{y_i}$, where $\|\varphi_i'\| = 1$. We want to show that for fixed x we have $y_1 = y_2$.

Suppose, to the contrary, that for some $x \in X$, we have $y_1 \neq y_2$. Then since U does not split, and $\{\varphi_1, \varphi_2\}$ is a basis for U^*, by [1, Lemma 1, p. 93] there exists a T-set T in U and a $\varphi \in U^* = V^0$ with $\|\varphi\| = 1$, such that $\langle u, \varphi\rangle = \|u\|$ for all $u \in T$, and such that both of the sets $\{\varphi, \varphi_1\}$ and $\{\varphi, \varphi_2\}$ are linearly independent.

By Lemma 3, $\langle e, \varphi\rangle = \|e\|$ for all e belonging to some T-set T in E, so that by [1, Lemma 2], $A^*(\varphi \cdot \mu_x)$ is of the form $\varphi' \cdot \mu_y$ for some $\varphi' \in E^*$ with $\|\varphi'\| = 1$, and some $y \in Y$. Let I be the subset of $\{1, 2\}$ such that for $i \in I$, the support of $A^{*-1}(\varphi_i \cdot \mu_x)$ is equal to y. Then I is either empty or a set containing one element.

We wish to show that $\{\varphi', \varphi_i' \colon i \in I\}$ is a linearly independent subset of E^*. This is trivially true if $I = \varnothing$, so suppose that I is a singleton, $I = \{i_0\}$, and suppose that there exists a scalar λ such that $\varphi' = \lambda\varphi_{i_0}'$. Then

$$\varphi' \cdot \mu_y - \lambda\varphi_{i_0}' \cdot \mu_y = 0$$

in $C(Y, E)^*$, and hence

$$A^*(\varphi' \cdot \mu_y - \lambda\varphi_{i_0}' \cdot \mu_y) = \varphi \cdot \mu_x - \lambda\varphi_{i_0} \cdot \mu_x = 0$$

in $C(X, E)^*$. Thus $\varphi = \lambda\varphi_{i_0}$ in E^*, contradicting the fact that $\{\varphi, \varphi_{i_0}\}$ is a linearly independent set. Hence $\{\varphi', \varphi_i' \colon i \in I\}$ is linearly independent as claimed.

Thus we can take a vector $e \in E$ such that $\langle e, \varphi_i'\rangle = 0$, $i \in I$, but $\langle e, \varphi'\rangle \neq 0$. Let g be any element of $C(Y)$ with $g(y) \neq 0$, and such that the support of g is disjoint from the support (or supports) of $A^{*-1}(\varphi_i \cdot \mu_x)$ for $i \notin I$, and define $G \in C(Y, E)$ by $G(y') = g(y') \cdot e$, $y' \in Y$. Since $\varphi \in U^*$ and $\{\varphi_1, \varphi_2\}$ is a basis for U^*, there exist scalars λ_i such that $\varphi + \lambda_1\varphi_1 + \lambda_2\varphi_2 = 0$. Thus $\varphi \cdot \mu_x + \lambda_1\varphi_1 \cdot \mu_x + \lambda_2\varphi_2 \cdot \mu_x = 0$ in $C(X, E)^*$, and so

$$\begin{aligned}
0 &= \int A^{-1}(G)\,d(\varphi \cdot \mu_x + \lambda_1\varphi_1 \cdot \mu_x + \lambda_2\varphi_2 \cdot \mu_x) \\
&= \int G\,d\big(A^{*-1}(\varphi \cdot \mu_x + \lambda_1\varphi_1 \cdot \mu_x + \lambda_2\varphi_2 \cdot \mu_x)\big) \\
&= \int G\,d(\varphi' \cdot \mu_y) + \sum_{i \in I} \lambda_i \int G\,d(\varphi_i' \cdot \mu_y) \\
&= \langle G(y), \varphi'\rangle + \sum_{i \in I} \lambda_i \langle G(y), \varphi_i'\rangle \\
&= g(y)\langle e, \varphi'\rangle \neq 0.
\end{aligned}$$

This contradiction completes the proof of the lemma.

THEOREM. *Let* E *be a three-dimensional Banach space. Then* E *fails to have* *the weak Banach-Stone property if, and only if, E is isometric to l_3^∞.*

PROOF. The "if" part of the theorem has been established by Sundaresan, [6, p. 22]. Hence we assume that E is not isometric to l_3^∞ and show that E has the weak Banach-Stone property.

If E does not split, then E has the Banach-Stone property, and we are done. Thus suppose that E splits, $E = U \oplus V$, where U is two dimensional and V is one dimensional. Now U cannot split, for otherwise E would be isometric to l_3^∞, contrary to the hypothesis.

Let A be an isometry of $C(X, E)$ onto $C(Y, E)$, and let $\{\varphi_1, \varphi_2\}$ be a T-basis for $U^* = V^0 \subseteq E^*$. Then for $x \in X$, define $\tau(x) = y$ if the supports of $A^{*-1}(\varphi_i \cdot \mu_x)$ are equal to y, for $i = 1, 2$. By Lemma 4, τ is a well-defined function from X to Y.

We wish to show, first of all, that τ is one-one. Suppose, to the contrary, that x_1, x_2 are distinct points of X, but that $\tau(x_1) = \tau(x_2) = y$. This would mean that $A^{*-1}(\varphi_i \cdot \mu_{x_j})$ is of the form $\varphi_{ij} \cdot \mu_y$ for $i = 1, 2$ and $j = 1, 2$. Since E^* is three dimensional, $\{\varphi_{ij}\colon i = 1, 2 \text{ and } j = 1, 2\}$ is a linearly dependent set, so that there exist scalars α_{ij}, not all zero, such that $\sum_{i,j \in \{1,2\}} \alpha_{ij}\varphi_{ij} = 0$. Then $\sum_{i,j \in \{1,2\}} \alpha_{ij}\varphi_{ij} \cdot \mu_y = 0$ in $C(Y, E)^*$, so that

$$A^*\Big(\sum_{i,j \in \{1,2\}} \alpha_{ij}\varphi_{ij} \cdot \mu_y\Big) = \sum_{i,j \in \{1,2\}} \alpha_{ij}\varphi_i \cdot \mu_{x_j} = 0$$

in $C(X, E)^*$. We may suppose, without loss of generality, that $\alpha_{11} \neq 0$. Since φ_1 and φ_2 are linearly independent, there exists an $e \in E$ with $\langle e, \varphi_2\rangle = 0 \neq \langle e, \varphi_1\rangle$. Take $f \in C(X)$ with $f(x_2) = 0 \neq f(x_1)$, and define $F \in C(X, E)$ by $F(x') = f(x') \cdot e$, for $x' \in X$. We would then have

$$\begin{aligned} 0 &= \int F d\Big(\sum_{i,j \in \{1,2\}} \alpha_{ij}\varphi_i \cdot \mu_{x_j}\Big) \\ &= \alpha_{11}\langle F(x_1), \varphi_1\rangle + \alpha_{21}\langle F(x_1), \varphi_2\rangle + 0 \\ &= \alpha_{11} f(x_1)\langle e, \varphi_1\rangle \neq 0, \end{aligned}$$

which is absurd. Thus τ is one-one, as claimed.

We next show that τ maps X onto Y. As above, we assume the contrary is true, and arrive at a contradiction. Suppose that $y_1 \in Y - \tau(X)$. By Lemma 4 and [1, Lemma 2], with A^* replacing A^{*-1}, we would have $A^*(\varphi_j \cdot \mu_{y_1})$ is of the form $\psi_j \cdot \mu_x$, $j = 1, 2$, for some point $x \in X$ dependent on y_1. Now $\tau(x) = y_2$, where by our assumption, y_2 cannot be equal to y_1. This means that $A^{*-1}(\varphi_i \cdot \mu_x) = \varphi_i' \cdot \mu_{y_2}$, $i = 1, 2$, where the φ_i' are elements of norm one in E^*. Since $\{\varphi_1, \varphi_2, \psi_1, \psi_2\}$ is a linearly dependent set in E^*, there are scalars λ_i, α_j, not all zero, such that $\sum_{i,j \in \{1,2\}} \lambda_i\varphi_i + \alpha_j\psi_j = 0$, and thus

$$\sum_{i,j \in \{1,2\}} \lambda_i\varphi_i \cdot \mu_x + \alpha_j\psi_j \cdot \mu_x = 0$$

in $C(X, E)^*$. Hence

$$A^{*-1}\left(\sum_{i,j\in\{1,2\}}\lambda_i\varphi_i\cdot\mu_x+\alpha_j\psi_j\cdot\mu_x\right)$$
$$=\lambda_1\varphi_1'\cdot\mu_{y_2}+\lambda_2\varphi_2'\cdot\mu_{y_2}+\alpha_1\varphi_1\cdot\mu_{y_1}+\alpha_2\varphi_2\cdot\mu_{y_1}=0$$

in $C(Y,E)^*$. If we note that since $\varphi_1\cdot\mu_x$ and $\varphi_2\cdot\mu_x$ are linearly independent elements of $C(X,E)^*$, $\varphi_1'\cdot\mu_{y_2}$ and $\varphi_2'\cdot\mu_{y_2}$ are linearly independent in $C(Y,E)^*$, and thus φ_1' and φ_2' are linearly independent in E^*, a construction exactly analogous to that of the preceding paragraph then yields an element $G\in C(Y,E)$ such that

$$\int G\,d(\lambda_1\varphi_1'\cdot\mu_{y_2}+\lambda_2\varphi_2'\cdot\mu_{y_2}+\alpha_1\varphi_1\cdot\mu_{y_1}+\alpha_2\varphi_2\cdot\mu_{y_1})\neq 0.$$

This contradiction thus establishes that τ is onto.

Finally, we show that τ is continuous. Suppose, to the contrary, that there exists a net $\{x_\beta:\beta\in B\}$ in X such that $x_\beta\to x_0$, but that $y_\beta=\tau(x_\beta)\not\to\tau(x_0)=y_0$. Then there is a compact neighborhood N of y_0 such that for every $\beta_0\in B$, there is a $\beta\geqslant\beta_0$ such that y_β lies outside N. Now by [1, Lemma 2] and by the definition of τ, $A^{*-1}(\varphi_1\cdot\mu_{x_0})=\varphi_{1,0}\cdot\mu_{y_0}$ for some $\varphi_{1,0}\in E^*$ with $\|\varphi_{1,0}\|=1$. Fix $e_0\in E$ with $\|e_0\|=1$, such that $\langle e_0,\varphi_{1,0}\rangle=1$. Choose $g_0\in C(Y)$ with $1=\|g_0\|_\infty=g_0(y_0)$, and such that the support of g_0 is contained in N. Then define $G_0\in C(Y,E)$ by $G_0(y)=g_0(y)\cdot e_0$, for $y\in Y$. We have

$$\left\langle(A^{-1}(G_0))(x_0),\varphi_1\right\rangle=\int A^{-1}(G_0)\,d(\varphi_1\cdot\mu_{x_0})$$
$$=\int G_0\,d\left(A^{*-1}(\varphi_1\cdot\mu_{x_0})\right)=\int G_0\,d(\varphi_{1,0}\cdot\mu_{y_0})$$
$$=\left\langle G_0(y_0),\varphi_{1,0}\right\rangle=\left\langle e_0,\varphi_{1,0}\right\rangle=1.$$

Now since $x_\beta\to x_0$ and $\langle(A^{-1}(G_0))(x),\varphi_1\rangle$ is a continuous function of $x\in X$, there exists a $\beta_0\in B$ such that if $\beta\geqslant\beta_0$ then $\operatorname{Re}\langle(A^{-1}(G_0))(x_\beta),\varphi_1\rangle>\frac12$. Thus fix a $\beta\geqslant\beta_0$ such that $y_\beta=\tau(x_\beta)$ lies outside N. Again we have $A^{*-1}(\varphi_1\cdot\mu_{x_\beta})=\varphi_{1,\beta}\cdot\mu_{y_\beta}$, for some $\varphi_{1,\beta}\in E^*$ with $\|\varphi_{1,\beta}\|=1$. Choose an $e_\beta\in E$ with $\|e_\beta\|=1$ such that $\langle e_\beta,\varphi_{1,\beta}\rangle=1$. Then take an element g_β of $C(Y)$ with $1=\|g_\beta\|_\infty=g_\beta(y_\beta)$, and such that the support of g_β is disjoint from N. Define $G_\beta\in C(Y,E)$ by $G_\beta(y)=g_\beta(y)\cdot e_\beta$, for $y\in Y$. Now G_0 and G_β both have norm one and they have disjoint supports, so that $\|G_0+G_\beta\|_\infty=1$. However,

$$\left\|A^{-1}(G_0+G_\beta)\right\|_\infty\geqslant\left\|(A^{-1}(G_0))(x_\beta)+(A^{-1}(G_\beta))(x_\beta)\right\|$$
$$\geqslant\operatorname{Re}\left[\left\langle(A^{-1}(G_0))(x_\beta),\varphi_1\right\rangle+\left\langle(A^{-1}(G_\beta))(x_\beta),\varphi_1\right\rangle\right]$$
$$>\frac12+\operatorname{Re}\int A^{-1}(G_\beta)\,d(\varphi_1\cdot\mu_{x_\beta})$$
$$=\frac12+\operatorname{Re}\int G_\beta\,d\left(A^{*-1}(\varphi_1\cdot\mu_{x_\beta})\right)=\frac12+\operatorname{Re}\int G_\beta\,d(\varphi_{1,\beta}\cdot\mu_{y_\beta})$$
$$=\frac12+\operatorname{Re}\left\langle G(y_\beta),\varphi_{1,\beta}\right\rangle=\frac12+\operatorname{Re}\left\langle e_\beta,\varphi_{1,\beta}\right\rangle=\frac32,$$

which contradicts the fact that A^{-1} is norm-preserving. Hence τ is a continuous, one-one map of X onto Y, and is thus a homeomorphism.

REFERENCES

1. M. Cambern, *On mappings of spaces of functions with values in a Banach space*, Duke Math. J. **42** (1975), 91–98.

2. ______, *Reflexive spaces with the Banach-Stone property*, Rev. Roumaine Math. Pures Appl. (to appear).

3. N. Dinculeanu, *Vector measures*, Pergamon Press, New York, 1967.

4. M. Jerison, *The space of bounded maps into a Banach space*, Ann. of Math. (2) **52** (1950), 309–327.

5. S. B. Myers, *Banach spaces of continuous functions*, Ann. of Math. (2) **49** (1948), 132–140.

6. K. Sundaresan, *Spaces of continuous functions into a Banach space*, Studia Math. **48** (1973), 15–22.

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