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ON μ-SPACES AND k_R-SPACES

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ABSTRACT. In this paper it is proved that when X is a k_R-space then μX (the smallest subspace of βX containing X with the property that each of its bounded closed subsets is compact) also is a k_R-space; an example is given of a k_R-space X such that its Hewitt realcompactification, υX, is not a k_R-space. We show with an example that there is a non-k_R-space X such that υX and μX are k_R-spaces. Also we answer negatively a question posed by Buchwalter: Is μX the union of the closures in υX of the bounded subsets of X? Finally, without using the continuum hypothesis, we give an example of a locally compact space X of cardinality $\aleph_1$ such that υX is not a k-space.

Introduction. The topological spaces used here will always be completely regular Hausdorff spaces. If X is a topological space we write $C(X)$ for the ring of the continuous real-valued functions on X, and βX (resp. υX) for the Stone-Čech compactification (resp. Hewitt realcompactification) of X. A subset M of X is said to be bounded if $g|_M$ is bounded for all $g \in C(X)$. A space is said to be a μ-space if every closed bounded subset is compact. Realcompact spaces (closed subspaces of a product of real lines) and P-spaces (spaces in which every G_δ is open) are μ-spaces. Write μX for the smallest subspace of βX that contains X and is a μ-space. A real-valued function g on X is called k_R-continuous if $g|_K$ is continuous in K for all compact subsets K of X. A space such that every k_R-continuous function is continuous is called a k_R-space. The associated k_R-space of a space X, denoted by $k_R X$, will be X provided with the coarsest topology for which every k_R-continuous function on X is continuous. It is easy to see that $k_R X$ is a completely regular Hausdorff space.

Our work provides the solutions to the following questions:

(1) If X is a k_R-space, is μX a k_R-space?

(2) If $\mathfrak{B}$ is the family of all closed bounded subsets of X, does the relation $\mu X = \bigcup \{\overline{B}^{\upsilon X}: B \in \mathfrak{B}\}$ hold?

(3) If υX or μX is a k_R-space, is X a k_R-space?

(4) If X is a k_R-space, is υX a k_R-space?

(5) If X is a realcompact space, is $k_R X$ realcompact?[1]

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[1]This question has been proposed by M. Valdivia.

The answer to question (1) is affirmative. Moreover, if X is a k_R-space, it is proven that $\bigcup\{\overline{B}^{\upsilon X}: B \in \mathfrak{B}\}$ is also, and that using this fact one has shown that μX is a k_R-space. Question (2) has a negative answer; the example is given of a locally compact space X such that $X \neq \bigcup\{\overline{B}^{\upsilon X}: B \in \mathfrak{B}\} \neq \mu X$. Using an example of Comfort [3], it is shown that the answer to question (3) is negative. Supposing continuum hypothesis (CH), an infinite class of k_R-spaces X is constructed such that υX is not a k_R-space. Thus question (4) has a negative answer. Further, it is shown that for the spaces X constructed before, $k_R(\upsilon X)$ is not realcompact and therefore question (5) also has a negative answer.

Nachbin [12] and Shirota [16] give the following characterizations for $C_c(X)$ (the vector space $C(X)$ provided with the compact-open topology):

(NS_1) X is realcompact if and only if $C_c(X)$ is bornological.

(NS_2) X is a μ-space if and only if $C_c(X)$ is barrelled.

These characterizations provide examples of nonbornological barrelled spaces, answering a question posed by Bourbaki [1]. Warner [18] characterizes the k_R-spaces as those spaces X for which $C_c(X)$ is complete (W).

Question 5 is related to the following problem posed by Dieudonné [4]: If E is a bornological space, is the completion of E a bornological space? T. Kōmura and Y. Kōmura [7], supposing (CH), give an example of a bornological space whose completion is not bornological, using sequence spaces of Köthe [8]. Since De Wilde and Schmets have proven [19] that X is realcompact if and only if $C_c(X)$ is an ultrabornological space (inductive limit of Banach spaces), the examples of realcompact spaces X such that $k_R X$ is not realcompact provide ultrabornological spaces $C_c(X)$ such that the completion is not a bornological space. Thus, our solution to question (5) is another solution to the problem proposed by Dieudonné, in the context of spaces $C_c(X)$.

A space is said to be a k-space if each of its subsets which has closed intersection with each compact subset is itself closed.Evidently each k-space is a k_R-space, but Pták presents [15] an example (credited to Katětov) which shows that the converse implication can fail. Another (completely regular Hausdorff) example has been discovered by Noble [14]. In [3], Comfort gives an example of a locally compact space X whose cardinality is $\aleph_2$ such that υX is not a k-space and asks himself if there exists a space of cardinality $\aleph_1$ with the same properties as the former one. In [13] Negrepontis, supposing (CH), gives an example of a space with these properties. We give a different example from the former one, this one without (CH).

Question 1. If X is a topological space, let μX be the intersection of all subspaces of βX which contain X and are μ-spaces. Thus, μX is a μ-space such that $X \subset \mu X \subset \upsilon X$. It can be easily shown that X is a μ-space if and only if $X = \mu X$, and that X is compact if and only if it is a pseudocompact μ-space. The space μX is unique in the following sense: If T is a μ-space which contains X as a dense subspace and every continuous mapping τ from

X into any μ-space Y has a continuous extension $\bar{\tau}$ from T into Y, then there exists a homeomorphism of μX onto T that leaves X pointwise fixed. Indeed, let ϕ be a continuous mapping from X into any μ-space Y; then ϕ has a Stone extension $\hat{\phi}$ to the whole βX into βY. If we prove that $E = \hat{\phi}^{-1}(Y)$ is a μ-space, then the restriction of $\hat{\phi}$ to μX is a continuous extension of ϕ because $\mu X \subset E$. If A is a bounded subset of E, then $B = \overline{\hat{\phi}(A)}$ is a compact subset of Y and since $\hat{\phi}^{-1}(B)$ is compact, the subset A is relatively compact in E. Therefore every continuous mapping from X into any μ-space Y has a continuous extension from μX into Y, and according to [5, 0.12] the proof is complete.

Proposition 1. *Let X be a topological space, let $\mathfrak{B}$ be the family of all bounded subsets of X, and let $E(X) = \bigcup\{\bar{B}^{vX}: B \in \mathfrak{B}\}$. If X is a k_R-space, then so is $E(X)$.*

Proof. If f is a k_R-continuous function in $E(X)$, and g is the restriction of f to X, then $g \in C(X)$. If h is the continuous extension of g to $E(X)$ and $x \in E(X) \sim X$, then $x \in \bar{A}^{vX}$, $A \in \mathfrak{B}$, since $K = \bar{A}^{vX} = \bar{A}^{E(X)}$ is compact, $f(x) = h(x)$ and $f = h$.

If α is an ordinal, we write $W(\alpha)$ for the set of all ordinals less than α. If M is a set, we denote the cardinal of M by $|M|$.

Theorem 1. *If X is a k_R-space, then so is μX.*

Proof. If $\aleph_\alpha$ is a cardinal larger than $2^{|\mu X|}$, let ω_α be the first ordinal whose cardinal is $\aleph_\alpha$. We define inductively $\{B_\sigma: \sigma \in W(\omega_\alpha)\}$, where $B_1 = E(X)$ and $B_\sigma = E(\bigcup\{B_\delta: \delta < \sigma\})$. Let us suppose that B_δ is a k_R-space for every $\delta < \sigma$, $\sigma \in W(\omega_\alpha)$ and we shall prove that B_σ is a k_R-space. By Proposition 1 it suffices to prove that $W = \bigcup\{B_\delta: \delta < \sigma\}$ is a k_R-space. If f is a k_R-continuous function in W, let g be the continuous extension to W of the restriction of f to X. If $x \in B_{\delta_0}$, $\delta_0 < \sigma$, since g and f are continuous in B_{δ_0} and coincide over X, it follows that $f(x) = g(x)$ and therefore, $g = f$. Thus, f is continuous in W and so W is a k_R-space. Let us now suppose that $\mu X \neq B_\sigma$ for every $\sigma \in W(\omega_\alpha)$, and we choose $\sigma_1 \in W(\omega_\alpha)$ such that $|\sigma_1| > |\mu X|$ (we write $|\sigma_1|$ for the cardinal number of σ_1). For every $\gamma < \sigma_1$ we choose a point $x_\gamma \in B_\gamma \sim \bigcup\{B_\delta: \delta < \gamma\}$. Therefore $|B_{\sigma_1} \sim X| \geqslant |\sigma_1|$ because $|\sigma_1| = |\{x_\gamma, \gamma \in W(\sigma_1)\}|$ and $x_\gamma \in B_{\sigma_1} \sim X$ for all $\gamma < \sigma_1$. On the other hand, the relation $|\sigma_1| > |\mu X| \geqslant |\mu X \sim X| \geqslant |B_{\sigma_1} \sim X|$ holds, which is a contradiction, so there exists $\sigma_0 \in W(\omega_\alpha)$ such that $B_{\sigma_0} = \mu X$.

Question 2. First, we shall give an example of a locally compact space X such that $|X| = \aleph_1$ and that vX is not a k-space. If α is an ordinal, we write $\alpha + 1$ for the ordinal which follows it and ω_0 (resp. ω_1) for the first infinite (resp. uncountable) ordinal. Let Y be the product space $W(\omega_1 + 1) \times W(\omega_0 + 1)$. If (α, ω_0) is a point of Y where α is a limit ordinal ($\alpha < \omega_1$), let $\{\beta_n\}_{n=1}^\infty$ be a strictly increasing sequence in $W(\alpha)$ which converges to α. If $\gamma_n = \beta_{n+1}$, $\alpha_n = \gamma_n + 1$, $n = 1,2,\ldots,$ it follows that $\{\alpha_n\}_{n=1}^\infty$ is a strictly increasing sequence which converges to α, α_n being an isolated point of $W(\omega_1)$. If p is a positive integer we write $A_{p,\alpha} = \{(\delta, p): \alpha_p \leqslant \delta \leqslant \alpha\}$ and $U_{n,\alpha} = \{(\alpha, \omega_0)\} \cup \{\bigcup_{p=n}^{\infty} A_{p,\alpha}\}$. If α is a limit ordinal of $W(\omega_1)$ let f_α be the function defined on Y as $f_\alpha(\alpha, \omega_0) = 0, f_\alpha(A_{n,\alpha}) = \{1/n\}$, $n = 1,2, \ldots,$ and, otherwise, as 1. Let $\mathfrak{F}$ be the weak topology on Y associated to $C(Y)$ and to the family of functions $\{f_\alpha, \alpha$ limit ordinal of $W(\omega_1)\}$. Then $(Y, \mathfrak{F})$ is a nonpseudocompact completely regular space, because $\{(\gamma_n, n)\}_{n=1}^{\infty}$ is a copy of N (discrete space of positive integers), which is C-embedded in Y. For the topology $\mathfrak{F}$ a basis of the neighborhoods of $(\eta, \sigma) \in Y$, when $\sigma < \omega_0$, is the family of the neighborhoods of this point in the product topology. The same is true for (ω_1, ω_0) and (η, ω_0) when η is a nonlimit ordinal of $W(\omega_1)$. If η is a limit ordinal of $W(\omega_1)$, a basis of the neighborhoods of (η, ω_0) is the family $\{U_{n,\eta}$: $n = 1,2, \ldots\}$. If $X = Y \sim \{(\omega_1, \omega_0)\}$, let us see that X is locally compact. If η is a limit ordinal of $W(\omega_1)$ and $\{V_i, i \in L\}$ is an open cover of $U_{1,\eta}$ there exists $n_0 \in N$, $i_0 \in L$ such that $U_{n_0,\eta} \subset V_{i_0}$. If $\{V_{i_j}: 1 \leqslant j \leqslant K\}$ is a finite subcover of the compact set $\bigcup \{A_{p,\eta}: 1 \leqslant p \leqslant n_0\}$, then $\{V_{i_j}: 0 \leqslant j \leqslant K\}$ is a finite subcover of $U_{1,\eta}$. Now we shall prove that X is C-embedded in Y. If $f \in C(X)$, there exists $\gamma \in W(\omega_1)$ such that if $\beta \geqslant \gamma$, then $f(\beta, n) = f(\omega_1, n)$, $n = 1,2, \ldots.$ By continuity, it results that $f(\beta, \omega_0) = f(\gamma, \omega_0)$ if $\beta \geqslant \gamma$, $\beta < \omega_1$. If $\hat{f}$ is the function that coincides on X with f and $\hat{f}(\omega_1, \omega_0) = f(\gamma, \omega_0)$ it follows that $\hat{f}$ is a continuous extension of f. If we prove that Y is realcompact, we shall have $Y = \upsilon X$. Let us suppose that M is a free real maximal ideal of $C(Y)$.[2] Then $M \neq \{f \in C(Y): f(\omega_1, \omega_0) = 0\}$ and if $Z(M) = \{Z(g): g \in M\}$ there will be $Z^1 \in Z(M)$, $\sigma_0 \in W(\omega_1)$ such that $Z^1 \cap \{(\beta, \omega_0): \sigma_0 + 1 \leqslant \beta \leqslant \omega_1\} = \emptyset$. Since $H_n = \{(\alpha, n): 1 \leqslant \alpha \leqslant \omega_1\}$ is a compact zero-set, then $H_n \notin Z(M)$. Thus there exists $Z_n \in Z(M)$ such that $Z_n \cap H_n = \emptyset$, $n = 1,2, \ldots.$ Since M is real, if $Z^2 = \bigcap_{n=1}^{\infty} Z_n$, then $Z^2 \in Z(M)$ and, therefore, $Z = Z^1 \cap Z^2 \in Z(M)$, $Z \subset \{(\beta, \omega_0): 1 \leqslant \beta \leqslant \sigma_0\}$, and so $Z = \{(\gamma_K, \omega_0)\}_{K=1}^{\infty}$, $1 \leqslant \gamma_1 < \gamma_2 < \cdots \leqslant \sigma_0$. Because $\{(\gamma_K, \omega_0)\}$ is a zero-set in Y, $K = 1,2, \ldots,$ and since M is real, there exists $K_0 \in N$ such that $\{(\gamma_{K_0}, \omega_0)\} \in Z(M)$ and, therefore, M is not free. This contradiction shows us that no free real maximal ideals in $C(Y)$ exist, and that, therefore, Y is realcompact. As the set $\{(\sigma, \omega_0): 1 \leqslant \sigma < \omega_1\}$ meets the compact subsets of Y in closed sets but is not closed, it results that Y is not a k-space.

We are now going to resolve negatively question (2) with an example. Let T be the subspace $Y \sim \{(\omega_1, n): 1 \leqslant n \leqslant \omega_0\}$, which is locally compact, and we shall prove that $T \neq \bigcup \{\bar{B}^{\upsilon T}: B \in \mathfrak{B}\} \neq \mu T$, $\mathfrak{B}$ being the family of all closed bounded sets of T. Since T is C-embedded in Y it follows that $Y = \upsilon T = \upsilon X$ and so $\mu T \subset \mu X$. The equality $X = \bigcup \{\bar{B}^{\upsilon T}: B \in \mathfrak{B}\}$ is a direct consequence of the following lemmas.

[2] We say that I is an ideal of $C(X)$ if it is a subring of $C(X)$ and if $f \in I$, $g \in C(X)$ implies $gf \in I$. An ideal I of $C(X)$ is said to be free when $\bigcap \{Z(f): f \in I\} = \emptyset$. A space X is realcompact if for every free maximal ideal I of $C(X)$, the residue class ring $C(X)/I$ is not isomorphic with the ring of the real numbers.

LEMMA 1. *If $A \subset T$ and $\{\sigma_K\}_{K=1}^{\infty}$ is a strictly increasing sequence in $W(\omega_1)$ such that $(\sigma_K, \omega_0) \in A$, $K = 1,2, \ldots,$ then A is not bounded in T.*

PROOF. If $\eta \in W(\omega_1)$ is not a limit ordinal we write $B(K, n_K) = \{(\eta,n): n_K \leqslant n \leqslant \omega_0\}$, and if it is a limit ordinal then $B(K,n_K) = U_{n_K,\eta}$. We define inductively a neighborhood $B(K,n_K)$ of (σ_K, ω_0) satisfying $B(K, n_K) \cap U_{1,\gamma} = \varnothing$ where $\gamma = \sup_K \sigma_K$ and for $1 \leqslant j \leqslant K-1$, $B(j, n_j) \cap B(K, n_K) = \varnothing$, $n_1 < n_2 < \cdots < n_K$. The function whose value is K in $B(K, n_K)$, $K = 1,2, \ldots,$ and vanishes otherwise, is continuous in T and nonbounded in A.

LEMMA 2. *If A is a closed set in T and $(\omega_1, \omega_0) \in \bar{A}^Y$, then A is not bounded in T.*

PROOF. Since $(\omega_1, \omega_0) \in \bar{A}^Y$ it is possible to choose a sequence $\{(\gamma_K^1, n_K^1)\}_{K=1}^{\infty}$ in A such that $n_K^1 < n_{K+1}^1, \gamma_K^1 < \gamma_{K+1}^1$, $K = 1,2, \ldots.$ With the product topology this sequence converges to (σ_1, ω_0), where $\sigma_1 = \sup_K \gamma_K^1$. If this sequence does not converge with the topology $\mathfrak{F}$ to (σ_1, ω_0), then it is a discrete closed set C-embedded in T contained in A. Therefore A is not bounded in T and the lemma is proved. If the sequence converges to (σ_1, ω_0) in the topology $\mathfrak{F}$, we can consider a sequence $\{(\gamma_K^2, n_K^2)\}_{K=1}^{\infty}$ in A satisfying $n_K^2 < n_{K+1}^2$, $\gamma_{K+1}^2 > \gamma_K^2 > \sigma_1$, $K = 1,2, \ldots,$ and we shall proceed as before. If for $p = 1,2, \ldots,$ the sequence $\{(\gamma_K^p, n_K^p)\}_{K=1}^{\infty}$ converges to (σ_p, ω_0) with the topology $\mathfrak{F}$, then $(\sigma_p, \omega_0) \in A$, $\sigma_p < \sigma_{p+1}$, $p = 1,2, \ldots,$ and therefore, from Lemma 1, the set A is not bounded in T.

Returning to our example, from Lemma 2 we deduce that $X = \bigcup \{\bar{B}^{vT}, B \in \mathfrak{B}\} \subset \mu T$, so that $\mu X \subset \mu T$ and consequently $\mu X = \mu T$. Thus, $X \neq \mu X$ since the set $\{(\omega_1, n): 1 \leqslant n < \omega_0\}$ is closed and bounded in X and noncompact.

Note. Since $Y = \mu X = vX$, by Proposition 1 it results that Y is a k_R-space. In [17] it is proved that kY[3] is not a regular space.

This example provides a solution to problem 2 of Buchwalter [2].

Question 3. Now we give an example of a topological space X such that $\mu X = vX$ is a k_R-space and X is not a k_R-space. Comfort [3] gives an example of a pseudocompact space X whose cardinality is c such that $N \subset X \subset \beta N$. Since every infinite closed set in $\beta N \sim N$ has cardinality 2^c [5, 9.12], it follows that the compact subsets of X are finite. But X is not discrete, so that it is not a k_R-space and $\mu X = vX = \beta X$.

Question 4. Firstly, let us note that every quotient space of a k_R-space is a k_R-space. This result is intimately connected with the fact that k_R is a coreflector (see [6] or [10]). Thus, if a topological product is a k_R-space then each factor space also is.

A subset of a topological space X is said to be k-closed if it intersects every

[3]If X is a topological space, the associated k-space to X, denoted by kX, will be X provided with the topology for which a set is closed if and only if it intersects every compact set of X in a closed set.

compact subset of X in a closed set of X. Therefore, a k-space is a space in which every k-closed subset is closed.

PROPOSITION 2. *If X is a μ-space and $\upsilon X \sim X$ is k-closed in υX, then υX is not a k_R-space.*

PROOF. Since X is a μ-space C-embedded in υX, then X is k-closed in υX. Moreover $\upsilon X \sim X$ is k-closed in υX. Therefore, the function whose value is 0 on X and 1 on $\upsilon X \sim X$ is k_R-continuous but is not continuous in υX.

THEOREM 2. *Let $\{U_i\}_{i \in I}$ be a family of clopen pairwise disjoint sets in $\beta N \sim N$ and let $X = N \cup \{\bigcup_{i \in I} U_i\}$. If $|\upsilon X \sim X| < 2^c$ and the points of $\upsilon X \sim X$ are not adherent to any countable subset in $X \sim N$, then X is a locally compact μ-space and υX is not a k_R-space.*

PROOF. As $\beta X = \beta N$ and X is open in βX, it results that X is locally compact. Let us see that X is a μ-space. Suppose that there is a noncompact closed bounded subset A in X. Since υX is a μ-space it follows that $K = \overline{A}^{\upsilon X}$ is compact. If A intersects infinitely many U_i we can choose a sequence $x_n \in A \cap U_{i_n}$, $n = 1,2, \ldots,$ with $i_n \neq i_m$ if $n \neq m$, such that $\{x_n\}_{n=1}^{\infty}$ has no adherent points in N and $X \sim N$ because U_i is open in $X \sim N$ for every $i \in I$ and $U_i \cap U_j = \emptyset$, $i \neq j$. Further, this sequence has no adherent points in $\upsilon X \sim X$ by hypothesis, and therefore K is not compact. Thus, there is a finite sequence $i_1, i_2, \ldots, i_n$, $i_j \in I$, $1 \leqslant j \leqslant n$, such that if $i \neq i_j$, $1 \leqslant j \leqslant n$, then $A \cap U_i = \emptyset$. Let W be a closed neighborhood of $\upsilon X \sim X$ such that $W \cap U_{i_j} = \emptyset$, $1 \leqslant j \leqslant n$. Then $A \cap N \cap W$ is a nonempty bounded subset in X and $K_0 = \overline{A \cap N \cap W}^{\upsilon X}$ is compact. So $K_0 \cap U_i = \emptyset$ for every $i \in I$ and, therefore, $|K_0| < 2^c$, which is a contradiction since the infinite compact subsets of βN have cardinality equal to 2^c. Thus every bounded closed subset of X is compact. On the other hand, $\upsilon X \sim X$ is closed in υX and from Proposition 2 we have that υX is not a k_R-space.

A point $x \in X$ is a P-point in X if $Z(f)$ is a neighborhood of x for all $f \in C(X)$ such that $f(x) = 0$. Then, X is a P-space if and only if every point is a P-point in X.

Let us now look at an example of a space satisfying the hypothesis of Theorem 2. Assume (CH). According to Rudin [5, 6V] there is a P-point p in $\beta N \sim N$. Let $\{G_\alpha\}_{\alpha < \omega_1}$ be a basis of clopen neighborhoods for p, such that $G_\alpha \subset G_\beta$, $G_\alpha \neq G_\beta$ for all $\beta < \alpha$. We define inductively a family of nonempty clopen sets $\{V_\alpha\}_{\alpha < \omega_1}$ in $\beta N \sim N$ such that $V_\alpha \subset G_\alpha \sim G_{\alpha+1}$ for all $\alpha < \omega_1$. If $X = N \cup \{\bigcup_{\alpha < \omega_1} V_\alpha\}$, according to Negrepontis [13], $\upsilon X = X \cup \{p\}$ and since p is a P-point in $\beta N \sim N$, it is not adherent to any sequence in $X \sim N$. From Theorem 2 we deduce that X is a k_R-space and that υX is not, with which question (4) is negatively resolved.

From this example we shall give an infinite class of k_R-spaces Z for which υZ is not a k_R-space. By a $\{0,1\}$-valued measure on a set F, we mean a countably additive function defined on the family of all subsets of F and assuming only the values 0 or 1. We call a cardinal $\mathfrak{m}$ measurable if a set F of cardinal $\mathfrak{m}$ admits a $\{0,1\}$-valued measure σ such that $\sigma(F) = 1$, and $\sigma(\{x\}) = 0$ for every $x \in F$. A discrete space is realcompact if and only if its cardinal is nonmeasurable [5, T.12.2].

THEOREM 3. *Let X be a locally compact space such that vX is not a k_R-space. Let Y be a pseudocompact k-space and suppose that $X \times Y$ has nonmeasurable cardinal. Then $\mu(X \times Y)$ is a k_R-space and $v(X \times Y)$ is not.*

PROOF. The locally compact spaces are characterized by the property that the products with k-spaces are k-spaces [11, T.3.1]. Therefore $X \times Y$ is a k-space and from Theorem 1, $\mu(X \times Y)$ is a k_R-space. On the other hand, $v(X \times Y) = vX \times vY$ [3, T.2.4], and since vX is not a k_R-space, it follows that $v(X \times Y)$ is not a k_R-space.

From (W), (NS_1) and (NS_2), we now obtain the following corollary.

COROLLARY. *Assuming* (CH), *there exist infinite spaces Z for which $C_c(Z)$ is a complete, barrelled, and nonbornological space.*

Note. If X is a discrete space of measurable cardinal then vX is a nondiscrete P-space and, therefore, vX is not a k_R-space.

Question 5. The main result which we shall now prove is that for k_R-spaces X the condition that vX is not a k_R-space is equivalent to the fact that $k_R(vX)$ is not realcompact. In [9] the following theorem is proved:

THEOREM A. *Let $(X,\mathfrak{T})$ be a completely regular Hausdorff space and let Y be a subset in βX which strictly contains X. Let $\mathfrak{T}_0$ be a topology on Y strictly finer than the induced topology by βX such that the restriction of $\mathfrak{T}_0$ to X coincides with $\mathfrak{T}$ and that X is a dense subset in Y for $\mathfrak{T}_0$. Then $(Y,\mathfrak{T}_0)$ is not a completely regular space.*

If X is a topological space provided with the topology $\mathfrak{T}$ and M is a subset of X, we denote by $M[\mathfrak{T}]$ the set M provided with the topology induced by $\mathfrak{T}$.

THEOREM 4. *If M is a k_R-space, then the following conditions are equivalent*:

(a) *vM is not a k_R-space.*

(b) *The associated k_R-space to vM is not realcompact.*

PROOF. That (b) implies (a) is trivial. We are going to prove that (a) implies (b). Write $\mathfrak{T}$ (resp. $\mathfrak{U}$) for the topology of $X = vM$ (resp. $k_R X$). Suppose that X is not a k_R-space. Since $X \neq k_R X$ we have that $\mathfrak{U}$ is strictly finer than $\mathfrak{T}$. Thus, since $M[\mathfrak{T}]$ is a k_R-space, it follows that both topologies coincide on M. According to Theorem A, M is not dense in $k_R X$. Suppose that $k_R X$ is realcompact and let $H = \overline{M}^{\mathfrak{U}}$. Then $X \neq H$ and $H[\mathfrak{U}]$ is realcompact. Let us now see that $C_{\mathfrak{U}}(H) \subset C_{\mathfrak{T}}(H)$, where $C_{\mathfrak{T}}(H)$ (resp. $C_{\mathfrak{U}}(H)$) is the ring of all continuous real-valued functions on $H[\mathfrak{T}]$ (resp. $H[\mathfrak{U}]$). If $f \in C_{\mathfrak{U}}(H)$ and $g = f|_M$ then $g \in C(M)$ and there is an extension $\hat{g} \in C_{\mathfrak{T}}(H)$ of g to H, because M is C-embedded in X. Thus, $\hat{g} \in C_{\mathfrak{U}}(H)$ and $\hat{g}|_M = f|_M$ and therefore $f = \hat{g}$ and $f \in C_{\mathfrak{F}}(H)$. From here it results that $C_{\mathfrak{U}}(H) = C_{\mathfrak{F}}(H)$ and that therefore $H[\mathfrak{F}]$ is realcompact, which is impossible, since $X = \upsilon M$ and $X \neq H$. Then $k_R X$ is not realcompact.

As a consequence of Theorem 3 we have the following

COROLLARY. *Let X be a locally compact space such that υX is not a k_R-space and let Y be a pseudocompact k-space. If $X \times Y$ has nonmeasurable cardinal, then $k_R(\upsilon(X \times Y))$ is not realcompact.*

Note. If X is a discrete space of measurable cardinal we know that υX is not a k_R-space and, according to Theorem 4, $k_R(\upsilon X)$ is not realcompact.

I am informed that Question 2 was also proven by R. Haydon with a different example.

REFERENCES

1. N. Bourbaki, *Sur certain espaces vectoriels topologiques*, Ann. Inst. Fourier (Grenoble) **2** (1950), 5–16. MR **13**, 137.

2. H. Buchwalter, *Parties bornées d'un espace topologique complètement régulier*, Séminaire Choquet: 1969/70, Initiation à l'Analyse, Fasc. 2, Exp. 14, Secrétariat Mathématique, Paris, 1970, pp. 1–15. MR **44** #3282.

3. W. W. Comfort, *On the Hewitt realcompactification of a product space*, Trans. Amer. Math. Soc. **131** (1968), 107–118. MR **36** #5896.

4. J. Dieudonné, *Recent developments in the theory of locally convex vector spaces*, Bull. Amer. Math. Soc. **59** (1953), 495–512. MR **15**, 963.

5. L. Gillman and M. Jerison, *Rings of continuous functions*, Van Nostrand, Princeton, N. J., 1960. MR **22** #6994.

6. J. F. Kennison, *Reflective functors in general topology and elsewhere*, Trans. Amer. Math. Soc. **118** (1965), 303–315. MR **30** #4812.

7. T. Kōmura and Y. Kōmura, *Sur les espaces parfaits de suites et leurs généralizations*, J. Math. Soc. Japan **15** (1963), 317–338. MR **28** #451.

8. G. Köthe, *Topological vector spaces*. I, Springer-Verlag, Berlin and New York, 1969. MR **40** #1750.

9. M. Lopez Pellicer, *Sobre ciertos espacios topológicos no completamente regulares*, Rev. Mat. Hisp.-Amer. **36** (1976), 125–132.

10. S. Mac Lane, *Categories for the working mathematician*, Graduate Texts in Math., vol. 5, Springer-Verlag, Berlin and New York, 1971. MR **50** #7275.

11. E. Michael, *Local compactness and Cartesian products of quotient maps and k-spaces*, Ann. Inst. Fourier (Grenoble) **18** (1968), 281–286. MR **39** #6256.

12. L. Nachbin, *Topological vector spaces of continuous functions*, Proc. Nat. Acad. Sci. U.S.A. **40** (1954), 471–474. MR **16**, 156.

13. S. Negrepontis, *An example of realcompactifications*, Arch. Math. (Basel) **20** (1969), 162–164. MR **39** #6265.

14. N. Noble, Doctoral dissertation, Univ. of Rochester, Rochester, N. Y., 1967.

15. V. Pták, *On complete topological linear spaces*, Czechoslovak Math. J. **3** (**78**) (1953), 301–364. MR **16**, 262.

16. T. Shirota, *On locally convex vector spaces of continuous functions*, Proc. Japan Acad. **30** (1954), 294–298. MR **16**, 275.

17. M. Valdivia, *On certain topologies on a vector space*, Manuscripta Math. **14** (1974), 241–247. MR **50** #14151.

18. S. Warner, *The topology of compact convergence on continuous functions*, Duke Math. J. **25** (1958), 265–282. MR **21** #1521.

19. M. de Wilde and J. Schmets, *Caractérisation des espaces $C(X)$ ultrabornologiques*, Bull. Soc. Roy. Sci. Liège **40** (1971), 119–121. MR **45** #872.

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