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LEFT AND RIGHT INVARIANCE IN AN INTEGRAL DOMAIN

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ABSTRACT. A ring is said to be right (left) invariant if each of its right (left) ideals is twosided. In this paper we resolve the conjecture: Every right invariant integral domain which satisfies the left Ore (multiple) condition is left invariant. A proof is given for the class of LCM domains satisfying a finiteness condition. An example is given to show that the LCM hypothesis cannot be dropped. A second example shows that the conjecture fails even in a Bezout domain which does not have the finiteness condition. The problem of right versus left boundedness is also considered.

An integral domain is said to be *right* (*left*) *invariant* if each of its right (left) ideals is twosided. This paper is motivated by the following question: [**3**, p. 162]: Is every right invariant integral domain which is assumed to be *left Ore* (intersection of any two nonzero left ideals is nonzero) also left invariant? We show that the answer is affirmative for LCM domains satisfying a finiteness condition, but is otherwise false. Other related questions (dealing with boundedness) are discussed.

In what follows R is an integral domain, i.e. a ring with unity which is free of proper divisors of zero. The definitions referred to above may be phrased completely in terms of principal ideals. In particular, if $Ra \subseteq aR$ then the element a of R is said to be *right invariant*; R is right invariant if each of its elements is. A similar statement holds for *left invariance*. If an element or a ring is both right and left invariant it is said to be *invariant*.

A *right* (*left*) *LCM domain* is a ring in which the intersection of any two principal right (left) ideals is again principal; an *LCM domain* is a ring that has both properties. If $0 \neq ab' = ba'$ in a right LCM domain R, then $aR \cap bR$ is generated by an element which is a least common right multiple of a and b, denoted $[a, b]_r$; in addition, the greatest common right factor $(a', b')_r$ of a' and b' exists. Similar remarks apply for left LCM domains. The following lemma, which is proved in [**1**], gives the relationship of these terms.

LEMMA 1. *Let R be an LCM domain and let $0 \neq ab' = ba' \in R$. Then*

$$ab' = ba' = [a, b]_r(a', b')_r = (a, b)_l[a', b']_l.$$

We shall also need the following easy result.

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LEMMA 2. *If* $0 \neq [a, b]_r = ba'$ *in a ring* R *and if* a *is right invariant, then* $aR \subseteq a'R$.

PROOF. By right invariance we have $ba \in aR \cap bR = ba'R$.

THEOREM 1. *Let* R *be an LCM domain which satisfies the ascending chain condition (acc) for either principal right or principal left ideals or which is atomic. If* R *is left Ore and right invariant then* R *is left invariant.*

PROOF. First we show that each *atom* (i.e. irreducible element) $a \in R$ is invariant. For any $r \in R$ let $xar = ya$ be a generator of $Rar \cap Ra$. The left Ore condition assures that this term is not zero so Lemma 1 applies and $(x, y)_l = 1$. If $d = (ar, a)_r$ then either $Rd = R$ or $Rd = Ra$ because a is an atom. In the first case we have $[x, y]_r = xar = ya$ by Lemma 1 and so $xR \subseteq aR$, $yR \subseteq arR \subseteq aR$ by Lemma 2 in contradiction with $(x, y)_l = 1$. Therefore $Rd = Ra$ and $ar \in Ra$. This shows that a is left invariant and hence invariant.

Next we show that R is *atomic*, i.e. each nonzero nonunit is a product of atoms. Suppose not. Using the left acc we may choose Rx maximal in $\{Rx | x$ is not a product of atoms$\}$, and choose an atom a such that $Rx \subseteq Ra$. Then $x \in aR$ (by invariance of a), i.e. $x = ax_1$ for some x_1 which cannot be a product of atoms; thus $Rx \subsetneq Rx_1$ is a contradiction. A similar argument shows that R is atomic if R satisfies the right acc. Each nonzero nonunit is a product of (invariant) atoms and is consequently invariant.

We single out the following special case of Theorem 1. Recall that a principal right ideal domain (PRI domain) has the acc for right ideals and is a right Ore domain; in addition, it is a weak Bezout domain (= 2-fir), and hence an LCM domain (cf. [3, pp. 47–50]). Theorem 1 and its left-right analog imply the following:

COROLLARY 1. *In a PRI domain the following two conditions are equivalent*:
(i) R *is left Ore and right invariant.*
(ii) R *is left invariant.*

We give two examples to show the necessity of the hypotheses in Theorem 1.

EXAMPLE 1 (A non-LCM domain with both chain conditions). Let F be a (commutative) field extension of a field K with an automorphism $\sigma: F \to F$ which maps K into a proper subfield of K. (For example, $F = Q(t_1, t_2, t_3, \ldots)$, $K = Q(t_1, t_3, t_5, \ldots)$, and σ the Q-automorphism of F defined by $\sigma(t_{2n-1}) = t_{2n+1}$, $\sigma(t_{2n+2}) = t_{2n}$ for $n = 1, 2, \ldots$ and $\sigma(t_2) = t_1$.)

Let P be the skew formal power series ring

$$P = F[[x, \sigma]] = \left\{ \sum_{i=0}^{\infty} x^i a_i | a_i \in F \right\}$$

in which multiplication is defined by $ax = xa^\sigma$ $(a \in F)$. It is easy to show that P is a local PRI and PLI domain (cf. [5]). Let $R = \{f(x) \in P | f(0) \in$

$K\}$ be the subring of P consisting of all power series with constant term in K. Clearly R is atomic; in fact, it is easily verified that R has the acc for both principal right and principal left ideals. In addition, R is left and right Ore because P has this property.

To show that R is right invariant we first observe that if h is any unit in P and $g \in R$ then $h^{-1}gh \in R$. Now let $f \in R$ be a nonunit written in the form $f = x^n h$ where $h(0) \neq 0$ (so that h is a unit in P) and $n > 0$. Clearly x is right invariant in R. Also, xh is right invariant, for if $r \in R$ and r' is chosen in R so that $rx = xr'$, then $rxh = xh(h^{-1}r'h) \in xhR$. Thus f is right invariant. Finally we note that x is not left invariant because $\sigma[K] \neq K$; therefore the only nonzero members of R that are left invariant are the units.

EXAMPLE 2 (A nonatomic Bezout domain). Let K be a commutative Bezout domain with quotient field F and with monomorphism $\sigma: K \to K$ which is not an epimorphism but which when extended to F in the usual way is an isomorphism. (For example we can take K to be the principal ideal domain of formal Laurent series $Z\langle\langle t\rangle\rangle$ over the ring of integers Z and take σ to be the Z-monomorphism defined by $\sigma(t) = 2t$.)

Let P be the ring of skew formal power series, as above, in which multiplication is defined by $ax = xa^\sigma$ $(a \in F)$. Let $R = \{f(x) \in P \mid f(0) \in K\}$. That R is a right and left Bezout domain follows from the proposition below.

We check the following.

(i) Each $a \in K$ is right and left invariant.

First observe that if $0 \neq a \in K$ and $f \in R$ then $a^{-1}fa$, $afa^{-1} \in R$. Thus $fa = a(a^{-1}fa) \in aR$ and, similarly, $af \in Ra$.

(ii) Each $f \in R$ is right invariant.

For f may be written $f = x^n uas^{-1}$ where u is a unit power series in R with $u(0) = 1$ and $a, s \in K$; then fs is right invariant in R (being a product of the right invariant elements x^n, u, a); since s is invariant it follows that f is right invariant.

(iii) The left invariant elements of R have the form $f = ub$, where $b \in K$ and u is a unit in R.

First observe that x is not left invariant because $\sigma[K] \neq K$. If $f = x^n uas^{-1}$ written as in (ii) is left invariant then so is $fs = x^n ua$ (because s is invariant); it then follows that x^n is left invariant because ua is invariant; contradiction with $\sigma[K] \neq K$ is avoided only if $n = 0$ and $f = uas^{-1}$. Since $u(0) = 1$ we have $as^{-1} \in K$ as desired.

PROPOSITION. *Let $P = F[[x, \sigma]]$ and $R = \{f(x) \in P \mid f(0) \in K\}$ be the rings constructed above. Then R is a right and left Bezout domain.*

PROOF. To show that R is right Bezout let $f, g \in R$. First assume that $f = a + f_1$, $g = b + g_1$ where $a, b \in K$ and $a \neq 0$. Let $dK = aK + bK$. Then since $f_1 = d(d^{-1}f_1)$, $g = d(d^{-1}g_1)$, we have $fR, gR \in dR$ so that $fR + gR \subseteq dR$. To show the reverse inclusion we observe that $fh = 1$ for some $h \in P$ and so $c = f(hc) \in fR$ for some $0 \neq c \in K$. Thus $a = f - f_1 = f - c(c^{-1}f_1) \in fR + gR$ and $b = g - g_1 = g - c(c^{-1}g_1) \in fR + gR$ which shows $dR \subseteq fR + gR$.

Now let $f = x^{n_1}(a_1b_1^{-1} + h_1) = x^{n_1}f_1$, $g = x^{n_2}(a_2b_2^{-1} + h_2) = x^{n_2}g_1$ where $h_i \in R$ and a_i, $b_i \neq 0$ in K. We assume that $0 < n_1 \leqslant n_2$. If $n_1 = n_2$ then $bf_1R + bg_1R = dR$, where $b = b_1b_2$ and $dK = a_1b_2K + a_2b_1K$. Multiplying on the left by $x^{n_1}b^{-1}$ we find $fR + gR = x^{n_1}b^{-1}dR$, a principal right ideal of R. If $n_1 < n_2$ then $g = x^{n_1}f_1(f_1^{-1}x^{n_2-n_1}g_2) \in fR$ so that $fR + gR = fR$.

We have shown that R is a right Bezout domain. Since P is a left and right Ore domain the same is true of R. Consequently, R is a left and right Bezout domain.

A ring is said to be *right* (*left*) *bounded* if each nonzero right (left) ideal contains a nonzero twosided ideal. Again the definition may be phrased in terms of principal one and twosided ideals. The example just given is a Bezout domain (hence left Ore) which is right invariant (hence right bounded) but not left bounded. The more difficult question of whether a ring satisfying one of the finiteness conditions of Theorem 1 which is left Ore and right bounded is also left bounded is open even for the special case of a PRI domain (however see Corollary 2 below). If R is a right bounded PLI domain then R is left bounded. For in this case a right invariant element $a \in R$ gives rise to a twosided ideal of the form aR which must be principal as a left ideal by hypothesis. From this it follows easily that $aR = Ra$ (cf. [**4**, p. 37]). In fact, in this case R is a PRI domain. We summarize in the following:

THEOREM 2. *A right bounded PLI domain is both left bounded and a PRI domain.*

PROOF. It remains to prove the second assertion. Since a right Ore PLI domain is a right Bezout domain it suffices to show that R is atomic. Let $a \in R$ be a nonzero nonunit and let a^* be its right bound (see [**2**] for the definition). We have noted above that all right invariant elements such as a^* must be invariant. Using the left acc we may write $a^* = a_1 \cdots a_n$, where each a_i is invariant but has no proper invariant factors. If p_i is an atomic factor of a_i, then $a_iR = p_i^*R$ where p_i^* is the right bound of p_i. By [**2**, Theorem 3.2] each p_i^* is a product of atoms (similar to p_i). Thus a^* and, consequently, a is a product of atoms.

COROLLARY 2. *Let R be an atomic weak Bezout domain. If R is left Ore and right bounded then R is left bounded.*

The converse in Corollary 2 (or in Theorem 1) does not hold as shown by the ring of skew formal power series $F[[x, \sigma]] = \{\sum_{i=0}^{\infty} a_ix^i \mid a_i \in F\}$ in which multiplication is defined by $xa = a^{\sigma}x$ for a monomorphism σ on the field F. This ring is a left invariant PLI domain which is not right bounded (indeed not right Ore) if σ is not an epimorphism.

We change sides and restate Theorem 2 for the sake of comparing with Corollary 1.

COROLLARY 3. *A left bounded PRI domain is a right bounded PLI domain.*

Whether "left bounded" may be replaced by the weaker "left Ore right bounded" in Corollary 3 is open. Indeed, whether the proposed hypothesis is actually weaker in a PRI domain is open. In an atomic PRI domain the two hypotheses are equivalent.

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