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GROUPS WITH AN ABELIAN SYLOW SUBGROUP

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ABSTRACT. The purpose of this note is to obtain a modular refinement of Brauer's induction theorem for groups with an abelian Sylow subgroup.

1. Introduction. Let χ be a complex irreducible character of a finite group G. Brauer's induction theorem shows that χ may be expressed as an integer linear combination of characters induced from elementary subgroups of G. The object of this note is to show that if a Sylow p-subgroup of G is abelian, then χ may be expressed as an integer linear combination of characters induced from elementary subgroups whose p-part is contained in the p-defect group of χ.

2. Notation. For H a finite group let $\mathrm{ch}(H)$, the ring of generalized characters, be the set of integer linear combinations of the complex irreducible characters of H. If χ and ψ are elements of $\mathrm{ch}(H)$, $(\chi,\psi)_H$ denotes their usual inner product.

A group E is elementary if it is the direct product of a q-group and a cyclic q'-group, for some prime q. Note that for any elementary group E, we may write $E = A \times B$ where A is a p-group and B is a p'-group. The subgroup A is referred to as the p-part of E.

For $g \in G$, let g_p denote the p-part of g. The p-defect group of an irreducible character is a defect group of the p-block which contains the character.

3. General case. Let P be a Sylow p-subgroup of G. In this section we assume that D is a subgroup of P which satisfies the following two conditions.

(i) For any $g \in G$, $D^g \cap P \subseteq D$.

(ii) For any subgroup $A \subseteq P$, the restriction homomorphism of $\mathrm{ch}(A)$ into $\mathrm{ch}(A \cap D)$ is surjective.

If $E = A \times B$ is an elementary subgroup of G, let $A' = A \cap D$ and $E' = A' \times B$.

PROPOSITION. *Let χ be an element of* $\mathrm{ch}(G)$ *such that $\chi(g) = 0$ if g_p is not conjugate to an element of D. Then χ may be expressed as an integer linear combination of characters induced from elementary subgroups whose p-part is contained in D.*

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Step 1. Let $E = A \times B$ be an elementary subgroup with $A \subseteq P$. Then $|E : E'|^{-1}\chi_{E'}$ is a generalized character of E'.

PROOF. It suffices to show that $|E : E'|^{-1}(\chi, \theta)_{E'}$ is an integer for any irreducible character θ of E'.

Now $A \subseteq P$ so by hypothesis (i), $a \in A$ is G-conjugate to an element of D if and only if $a \in A'$. Since χ vanishes on any element whose p-part is not conjugate to an element of D, it follows that χ_E vanishes on any element not in E'. So for $\psi \in \operatorname{ch}(E)$

$$(\chi, \psi)_E = |E : E'|^{-1}(\chi, \psi)_{E'}.$$

Any irreducible character of $A' \times B$ is of the form $\alpha \times \beta$, where α and β are irreducible characters of A' and B respectively. By hypothesis (ii), α is the restriction to A' of some generalized character of A. It follows that the restriction homomorphism of $\operatorname{ch}(E)$ into $\operatorname{ch}(E')$ is surjective. Choosing $\psi \in \operatorname{ch}(E)$ such that $\psi_{E'} = \theta$ in the preceding equation establishes the result.

Step 2. Let $E = A \times B$ be an elementary subgroup with $A \subseteq P$. If $\psi \in \operatorname{ch}(E)$, then

$$(\psi\chi_E)^G = |E : E'|^{-1}((\psi\chi)_{E'})^G.$$

PROOF. For $g \in G$ we have

$$(\psi\chi_E)^G(g) = |E|^{-1} \sum_{x \in G} \psi^o(g^x)\chi(g^x)$$

where ψ^o is defined by $\psi^o(y) = \psi(y)$ if $y \in E$ and 0 otherwise. Now if $y \in E$, then $y_p \in A$ and again by hypothesis (i), $\chi(y) = 0$ if $y_p \notin A'$. So

$$\chi\psi^o = \chi(\psi_{E'})^o$$

and the result then follows from the definition of the induced character.

Step 3. We now complete the proof of the proposition. By Brauer's induction theorem [2, Theorem 16.2] we may write

$$(*) \qquad 1 = \sum a_i \psi_i^G$$

where the a_i are integers and the ψ_i are linear characters of elementary subgroups E_i. Write $E_i = A_i \times B_i$. Replacing E_i by a conjugate subgroup if necessary, we may assume that $A_i \subseteq P$.

From (*) we have

$$\chi = \sum a_i (\psi_i \chi_{E_i})^G$$

and by Step 2

$$(\psi_i\chi_{E_i})^G = \theta_i^G, \quad \text{where } \theta_i = |E_i : E_i'|^{-1}(\psi_i\chi)_{E_i'}.$$

Now by Step 1, θ_i is a generalized character of E_i', which proves the proposition.

REMARK. In equation (*) it is easy to see that there is some elementary subgroup E_i with $A_i = P$. So the corresponding θ_i which appears in the expression for χ is a generalized character of an elementary subgroup whose p-part equals D.

4. Applications. We first apply the proposition with $D = 1$ to obtain the following result of Brauer [**1**, Theorem 5].

COROLLARY 1. *Suppose that* $\chi \in \mathrm{ch}(G)$ *is such that* $\chi(g) = 0$ *if* $g_p \neq 1$. *Then* χ *is an integer linear combination of characters induced from elementary subgroups whose order is prime to* p.

Note that if χ is a principal indecomposable character for the prime p, then χ satisfies the hypothesis of the corollary.

For D a p-subgroup of G, let $\mathrm{ch}_D(G)$ denote the elements of $\mathrm{ch}(G)$ which are integer linear combinations of characters of p-adic indecomposable representations with vertex contained in D. Note that if $\chi \in \mathrm{ch}_D(G)$, then χ vanishes on any element of G whose p-part is not conjugate to an element of D [**2**, Lemma 59.5]. Moreover, if χ is an irreducible character of G with p-defect group D, then a vertex for χ is contained in D and hence $\chi \in \mathrm{ch}_D(G)$. So the result stated in the introduction is a consequence of the following corollary.

COROLLARY 2. *Suppose that a Sylow* p*-subgroup of* G *is abelian. Let* D *be a* p*-subgroup of* G. *Then every element of* $\mathrm{ch}_D(G)$ *is an integer linear combination of characters induced from elementary subgroups whose* p*-part is contained in* D.

PROOF. We use induction on $|D|$. If $|D| = 1$, the result follows from Corollary 1.

In general, if $\chi \in \mathrm{ch}_D(G)$, then the Green correspondence [**3**, Theorem 2] applied to G, $N = N_G(D)$ and D shows that

$$\chi - \psi^G \in \sum_{g \notin N} \mathrm{ch}_{D \cap D^g}(G)$$

for some $\psi \in \mathrm{ch}_D(N)$. For $g \notin N$, the induction hypothesis yields the result for $\mathrm{ch}_{D \cap D^g}(G)$. Since a Sylow p-subgroup of N is abelian, D satisfies the two conditions of §3 with respect to the group N. Now applying the proposition to the group N and the character ψ yields the result for ψ and hence completes the proof of the corollary.

It would be interesting to know to what extent the analog of Corollary 2 holds for an arbitrary finite group.

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