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ON CONTRACTIONS SATISFYING Alg $T = \{T\}'$

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ABSTRACT. For a bounded linear operator T on a Hilbert space let $\{T\}'$, $\{T\}''$ and Alg T denote the commutant, the double commutant and the weakly closed algebra generated by T and 1, respectively. Assume that T is a completely nonunitary contraction with a scalar-valued characteristic function $\psi(\lambda)$. In this note we prove the equivalence of the following conditions: (i) $|\psi(e^{it})| = 1$ on a set of positive Lebesgue measure; (ii) Alg $T = \{T\}'$; (iii) every invariant subspace for T is hyperinvariant. This generalizes the well-known fact that compressions of the shift satisfy Alg $T = \{T\}'$.

For an arbitrary operator T on a Hilbert space it is easily seen that the inclusions Alg $T \subseteq \{T\}'' \subseteq \{T\}'$ hold. Let H^2 be the usual Hardy space and let ψ be a scalar-valued inner function. Consider the compression of the shift T defined on the space $H^2 \ominus \psi H^2$ by

$$(Tf)(e^{it}) = P\left[e^{it}f(e^{it})\right] \quad \text{for } f \in H^2 \ominus \psi H^2,$$

where P denotes the (orthogonal) projection onto the space $H^2 \ominus \psi H^2$. It was shown by Sarason [3] that Alg $T = \{T\}'$. (In fact, he showed more than this. He proved that every operator in $\{T\}'$ is of the form $U(T)$ for some $U \in H^\infty$.) Note that here T is a completely nonunitary (c.n.u.) contraction whose characteristic function ψ is scalar-valued and satisfies $|\psi(e^{it})| = 1$ a.e. In this note we give necessary and sufficient conditions that a c.n.u. contraction with a scalar-valued characteristic function satisfy Alg $T = \{T\}'$. Indeed, we want to prove

THEOREM. *Let T be a c.n.u. contraction with a scalar-valued characteristic function ψ. Then the following conditions are equivalent to each other*:

(i) $|\psi(e^{it})| = 1$ *on a set of positive Lebesgue measure*;

(ii) Alg $T = \{T\}'$;

(iii) *every invariant subspace for T is hyperinvariant.*

Thus Sarason's result follows from the implication (i) $\Rightarrow$ (ii) of our Theorem. It is interesting to contrast our result with the fact, due to Sz.-Nagy and Foiaş [6], that a c.n.u. contraction T with the scalar-valued characteristic function ψ satisfies $\{T\}'' = \{T\}'$ if and only if $\psi(\lambda) \not\equiv 0$. Note also that

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whether (ii) and (iii) are equivalent for an arbitrary operator T is still an open question (cf. [1]).

In the proof of our Theorem we will extensively use the functional model for c.n.u. contractions. The readers are referred to [5] for the basic definitions and terminologies. Throughout this note results from [5] will be used without specific mentioning.

Let T be a c.n.u. contraction with the scalar-valued characteristic function ψ. Consider the functional model for T, that is, consider T being defined on the space $H \equiv [H^2 \oplus \overline{\Delta L^2}] \ominus \{\psi w \oplus \Delta w \colon w \in H^2\}$ by

$$T(f \oplus g) = P(e^{it}f \oplus e^{it}g) \quad \text{for } f \oplus g \in H,$$

where $\Delta = (1 - |\psi|^2)^{1/2}$ and P denotes the (orthogonal) projection onto H. Let Lat T denote the lattice of invariant subspaces for T, and let $T^{(n)}$ denote the operator $\underbrace{T \oplus \cdots \oplus T}_{n}$ acting on the space $\underbrace{H \oplus \cdots \oplus H}_{n}$. Note that the characteristic function of $T^{(n)}$ is the $n \times n$ matrix-valued function

$$\Phi = \begin{bmatrix} \psi & & 0 \\ & \ddots & \\ 0 & & \psi \end{bmatrix}.$$

Let $K \in \text{Lat } T^{(n)}$ and let $\Phi = \Phi_2\Phi_1$ be the corresponding regular factorization. We first prove the following

Lemma. *If $|\psi(e^{it})| = 1$ on a set of positive Lebesgue measure, then Φ_1 and Φ_2 are $n \times n$ matrix-valued functions.*

Proof. Assume that Φ_1 and Φ_2 are, respectively, $m \times n$ and $n \times m$ matrix-valued functions. Let

$$\Delta(e^{it}) = \left(1 - \Phi(e^{it})^*\Phi(e^{it})\right)^{1/2}$$

and

$$\Delta_j(e^{it}) = \left(1 - \Phi_j(e^{it})^*\Phi_j(e^{it})\right)^{1/2}, \qquad j = 1, 2.$$

Let $\delta(e^{it}) = \dim \overline{\Delta(e^{it})\mathbf{C}^n}$, $\delta_1(e^{it}) = \dim \overline{\Delta_1(e^{it})\mathbf{C}^n}$ and $\delta_2(e^{it}) = \dim \overline{\Delta_2(e^{it})\mathbf{C}^m}$, where $\mathbf{C}$ denotes the complex plane. Since $\Phi = \Phi_2\Phi_1$ is a regular factorization, we have

$$\delta(e^{it}) = \delta_1(e^{it}) + \delta_2(e^{it}) \quad \text{a.e.} \tag{1}$$

(cf. [5, Proposition VII. 3.3]). Since $|\psi(e^{it})| = 1$ on a set of positive Lebesgue measure, say α, it follows that $\Delta(e^{it}) = 0$ on α. Hence $\delta(e^{it}) = 0$ on α. If $m > n$ then $\Phi_2(e^{it})$ cannot be isometric from $\mathbf{C}^m$ to $\mathbf{C}^n$. Thus $\delta_2(e^{it}) > 0$ a.e., which contradicts (1). On the other hand, if $m < n$, then $\Phi_1(e^{it})$ cannot be isometric from $\mathbf{C}^n$ to $\mathbf{C}^m$. Then $\delta_1(e^{it}) > 0$ a.e. and we also have a contradiction. This proves that $m = n$.

Proof of the Theorem. If $\psi \equiv 0$, then, by the previously mentioned result of Sz.-Nagy and Foiaş [6], it is easily seen that none of the three conditions is satisfied. Hence we may assume hereafter that $\psi \not\equiv 0$.

(i) $\Rightarrow$ (ii). Let S be an operator in $\{T\}'$. To show that $S \in \operatorname{Alg} T$ it suffices to show that $\operatorname{Lat} T^{(n)} \subseteq \operatorname{Lat} S^{(n)}$ for all $n \geq 1$ (cf. [2, Theorem 7.1]). Let $K \in \operatorname{Lat} T^{(n)}$ and $\Phi = \Phi_2\Phi_1$ be the corresponding regular factorization. As proved in the Lemma, Φ_1 and Φ_2 are $n \times n$ matrix-valued functions. In the functional model of $T^{(n)}$,

$$K = \left\{\Phi_2 u \oplus Z^{-1}(\Delta_2 u \oplus v) : u \in H^2(\mathbf{C}^n), v \in \overline{\Delta_1 L^2(\mathbf{C}^n)}\right\}$$
$$\ominus \left\{\Phi w \oplus \Delta w : w \in H^2(\mathbf{C}^n)\right\},$$

where Z denotes the unitary operator from $\overline{\Delta L^2(\mathbf{C}^n)}$ to $\overline{\Delta_2 L^2(\mathbf{C}^n)} \oplus \overline{\Delta_1 L^2(\mathbf{C}^n)}$ defined by

$$Z(\Delta v) = \Delta_2 \Phi_1 v \oplus \Delta_1 v, \qquad v \in L^2(\mathbf{C}^n).$$

Let $\Phi_2 u \oplus t$ be an element in K, where $u = (u_i)_i \in H^2(\mathbf{C}^n)$ and $t = (t_i)_i \in \overline{\Delta L^2(\mathbf{C}^n)}$ satisfy $Z(t) = \Delta_2 u \oplus v$ for some $v = (v_i)_i \in \overline{\Delta_1 L^2(\mathbf{C}^n)}$. Here we use the symbol $(\ )_i$ to denote the components of a vector. We want to show that $S^{(n)}(\Phi_2 u \oplus t) \in K$. Note that S is of the form

$$S = P\begin{pmatrix} A & 0 \\ B & C \end{pmatrix},$$

where $A \in H^\infty$ and $B, C \in L^\infty$ satisfy $B\psi + C\Delta = \Delta A$ a.e. (cf. [6]). Assume that $\Phi_1 = (\xi_{ij})$ and $\Phi_2 = (\psi_{ij})$. Since

$$\Phi_2 u \oplus t = \left(\sum_{j=1}^n \psi_{ij} u_j\right)_i \oplus (t_i)_i,$$

we have

$$S^{(n)}(\Phi_2 u \oplus t) = \left[P\begin{pmatrix} A & 0 \\ B & C\end{pmatrix}\begin{bmatrix} \sum_{j=1}^n \psi_{ij} u_j \\ t_i \end{bmatrix}_i\right]_i$$

$$= \left[P\begin{bmatrix} A\sum_{j=1}^n \psi_{ij} u_j \\ B\sum_{j=1}^n \psi_{ij} u_j + Ct_i \end{bmatrix}_i\right]_i \tag{2}$$

$$= \begin{bmatrix} A\sum_{j=1}^n \psi_{ij} u_j - \psi w_i \\ B\sum_{j=1}^n \psi_{ij} u_\gamma + Ct_i - \Delta w_i \end{bmatrix}_i,$$

for some $w_i \in H^2$, $i = 1, 2, \ldots, n$. Since $\Phi = \Phi_2\Phi_1$, we have

$$\sum_{k=1}^{n} \psi_{ik}\xi_{kj} = \begin{cases} \psi, & \text{if } j = i, \\ 0, & \text{otherwise,} \end{cases} \qquad i, j = 1, \ldots, n.$$

Using Cramer's rule to solve this system of equations for ψ_{ik}, we obtain

$$(\det \Phi_1)\psi_{ik} = \psi\eta_{ik} \qquad i, k = 1, \ldots, n,$$

where η_{ik} is the determinant, multiplied by $(-1)^{i+k}$, of the matrix obtained from Φ_1 by deleting its ith column and kth row. It follows that

$$(\det \Phi_1)B \sum_{j=1}^{n} \psi_{ij}u_\gamma = B \sum_{j=1}^{n} \psi\eta_{ij}u_j = \Delta(A - C) \sum_{j=1}^{n} \eta_{ij}u_j.$$

Hence $(\det \Phi_1)B(\sum_{j=1}^{n} \psi_{ij}u_j)_i$ is an element of $\overline{\Delta L^2(\mathbf{C}^n)}$. Thus we have

$$Z\left[(\det \Phi_1)B\left(\sum_{j=1}^{n} \psi_{ij}u_j\right)_i\right] = Z\left[\Delta(A - C)\left(\sum_{j=1}^{n} \eta_{ij}u_j\right)_i\right]$$

$$= \left[\Delta_2\Phi_1(A - C)\left(\sum_{j=1}^{n} \eta_{ij}u_j\right)_i\right] \oplus \left[\Delta_1(A - C)\left(\sum_{j=1}^{n} \eta_{ij}u_j\right)_i\right]$$

$$= \left[\Delta_2(A - C)\left[\sum_{k=1}^{n} \xi_{ik}\left(\sum_{j=1}^{n} \eta_{kj}u_j\right)\right]_i\right] \oplus \left[\Delta_1(A - C)\left(\sum_{j=1}^{n} \eta_{ij}u_j\right)_i\right].$$

Since

$$\sum_{k=1}^{n} \xi_{ik}\left(\sum_{j=1}^{n} \eta_{kj}u_j\right) = \sum_{j=1}^{n}\left(\sum_{k=1}^{n} \xi_{ik}\eta_{kj}\right)u_j$$

$$= \sum_{j=1}^{n} (\det \Phi_1)\delta_{ij}u_j \qquad (\delta_{ij} \text{ the Kronecker } \delta) = (\det \Phi_1)u_i,$$

the above becomes

$$\begin{aligned} &\left[\Delta_2(A - C)((\det \Phi_1)u_i)_i\right] \oplus \left[\Delta_1(A - C)\left(\sum_{j=1}^{n} \eta_{ij}u_j\right)_i\right] \\ &\qquad = \left[\Delta_2(A - C)(\det \Phi_1)u\right] \oplus \left[\Delta_1(A - C)\left(\sum_{j=1}^{n} \eta_{ij}u_j\right)_i\right]. \end{aligned} \tag{3}$$

On the other hand,

$$\begin{aligned} Z\left[(\det \Phi_1)B\left(\sum_{j=1}^{n} \psi_{ij}u_j\right)_i\right] &= (\det \Phi_1)Z\left[\left(B \sum_{j=1}^{n} \psi_{ij}u_j\right)_i\right] \\ &= (\det \Phi_1)(X \oplus Y), \end{aligned} \tag{4}$$

say, for some element $X \oplus Y$ in $\overline{\Delta_2 L^2(\mathbf{C}^n)} \oplus \overline{\Delta_1 L^2(\mathbf{C}^n)}$. Equating the first components in (3) and (4) we obtain

$$\Delta_2(A - C)(\det \Phi_1)u = (\det \Phi_1)X. \tag{5}$$

Since $\psi \not\equiv 0$, we have $\det \Phi \not\equiv 0$, and hence $\det \Phi_1 \not\equiv 0$. By the F. and M. Riesz theorem, (5) yields that $\Delta_2(A - C)u = X$. Thus

$$\begin{aligned} Z\left[\left(B\sum_{j=1}^{n} \psi_{ij}u_j + Ct_i\right)_i\right] &= Z\left[\left(B\sum_{j=1}^{n} \psi_{ij}u_j\right)_i\right] + Z((Ct_i)_i) \\ &= (X \oplus Y) + Z(Ct) = \left[\Delta_2(A - C)u \oplus Y\right] + C(\Delta_2 u \oplus v) \\ &= \Delta_2 Au \oplus (Y + Cv). \end{aligned}$$

Hence (2) can be written as

$$S^{(n)}(\Phi_2 u \oplus t) = \left\{\Phi_2 Au \oplus Z^{-1}\left[\Delta_2 Au \oplus (Y + Cv)\right]\right\} - (\Phi w \oplus \Delta w),$$

where $w = (w_i)_i \in H^2(\mathbf{C}^n)$. This shows that $S^{(n)}(\Phi_2 u \oplus t) \in K$ as asserted and completes the proof of the implication (i) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (iii). This is trivial.

(iii) $\Rightarrow$ (i). Assume $|\psi(e^{it})| < 1$ a.e. It was proved in [7] that the hyperinvariant subspaces for T are of the form $\{f \oplus g \in H: -\Delta f + \psi g \in L^2(E) \text{ and } f \in IH^2\}$, where E is a measurable subset of the unit circle and I is an inner divisor of ψ_i where ψ_i denotes the inner factor of ψ. By Proposition 7.2 of [4], invariant subspaces of this form are precisely those arising from scalar regular factorizations of ψ. However, since $|\psi(e^{it})| < 1$ a.e., it is known [5, p. 301] that nontrivial vector regular factorizations of ψ exist. By the uniqueness of the correspondence between regular factorizations of ψ and invariant subspaces for T, the invariant subspace corresponding to any such vector regular factorization of ψ cannot arise from a scalar regular factorization, and hence is not hyperinvariant. Thus we obtain a contradiction of (iii) and complete the proof.

COROLLARY. *Let T be a c.n.u. contraction with a scalar-valued inner characteristic function. Then* $\operatorname{Alg} T = \{T\}'$.

We are grateful to the referee for making the proof of (iii) $\Rightarrow$ (i) of our Theorem more conceptual and less computational.

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