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A TREE ARGUMENT IN INFINITARY MODEL THEORY[1]

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ABSTRACT. A tree argument is used to show that any counterexample to Vaught's conjecture must have an uncountable model. A similar argument replaces the use of forcing by Burgess in a theorem on Σ_1^1 equivalence relations.

A *formula* or a *sentence* is one of $L_{\omega_1\omega}$. A sentence ϕ is a *counterexample to Vaught's conjecture* or, simply, a *counterexample* if it has more than $\aleph_0$ but less than $2^{\aleph_0}$ nonisomorphic countable models. A sentence is *large* if it has more than $\aleph_0$ nonisomorphic countable models. A large sentence is *minimal* if for every sentence ψ, either $\phi \wedge \psi$ or $\phi \wedge \neg \psi$ is not large.

THEOREM 1. *Every counterexample can be strengthened to a minimal counterexample, i.e., if σ is a counterexample, then there is a minimal counterexample ϕ such that $\phi \models \sigma$.*

The proof of Theorem 1 uses a lemma due to Morley **[8]**, whose formulation depends on the following (stronger than usual) notion of *fragment*. A set of formulas Δ is a fragment if it is closed under subformulas, substitutions of terms, finitary logical operations and if it satisfies: whenever $\phi \in \Delta$, $\bigvee\Theta \in \Delta$ (where $\Theta \subset \Delta$), then $\bigvee\{\exists x\theta: \theta \in \Theta\}$, $\bigvee\{\phi \wedge \theta: \theta \in \Theta\}$, $\bigvee(\{\phi\} \cup \Theta)$ all belong to Δ.

LEMMA 2 **[8]**. *Let Δ be a countable fragment. If $T \subset \Delta$ is a finitely consistent set of sentences such that for all valid $\bigvee\Theta \in \Delta$ there is a $\theta \in \Theta$ which belongs to T, then T is consistent.*

PROOF OF THEOREM 1. Assume that the theorem is false for some σ. Then it is easily seen that there is a countable fragment Δ containing σ and such that for every large $\phi \in \Delta$ s.t. $\phi \models \sigma$, there is $\psi \in \Delta$ with both $\phi \wedge \psi$ and $\phi \wedge \neg \psi$ large. Let $\{\delta_n\}_{n<\omega}$ be an enumeration of *all* valid disjunctions belonging to Δ. We are going to define a *tree* T_s, $s \in {}^{<\omega}2$ (= the set of finite sequences of 0's and 1's) such that for all s:

(a) T_s is a finite subset of Δ, $\sigma \in T_s$ and $\bigwedge T_s$ is a large sentence;

(b) $T_{s^\wedge\langle 0\rangle}$ and $T_{s^\wedge\langle 1\rangle}$ are contradictory; and

(c) if $\mathrm{lh}\, s = i$ and $\delta_i = \bigvee\Theta$ then there are $\theta', \theta'' \in \Theta$ s.t. $\theta' \in T_{s^\wedge\langle 0\rangle}$ and $\theta'' \in T_{s^\wedge\langle 1\rangle}$.

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Once the tree is constructed, consider, for each $\eta \in {}^{\omega}2$ (= the set of infinite sequences of 0's and 1's), the set $T_\eta = \bigcup\{T_{\eta\restriction i}: i < \omega\}$. By (a), (c) and Lemma 2, each T_η is consistent, hence has a countable model M_η (which, by (a), is a model of σ); by (b), if $\eta \neq \eta'$ then $M_\eta \not\cong M_{\eta'}$. Hence $\{M_\eta: \eta \in {}^{\omega}2\}$ is a collection of $2^{\aleph_0}$ nonisomorphic countable models of σ, a contradiction to the assumption that σ is a counterexample.

Thus, to conclude the proof, we have to indicate how to get T_s. This is done by induction on the length of s. Take $T_{\langle\rangle} = \{\sigma\}$. Assume that T_s has been defined and lh $s = i$. As $\bigwedge T_s \in \Delta$ is large and implies σ, there is $\psi \in \Delta$ s.t. both $\bigwedge T_s \wedge \psi$ and $\bigwedge T_s \wedge \neg\psi$ are large. If $\delta_i = \bigvee\Theta$ then any of the uncountably many models of $\bigwedge T_s \wedge \psi$ ($\bigwedge T_s \wedge \neg\psi$) is a model of some $\bigwedge T_s \wedge \psi \wedge \theta$ ($\bigwedge T_s \wedge \neg\psi \wedge \theta$), with $\theta \in \Theta$. Thus, there is $\theta' \in \Theta$ ($\theta'' \in \Theta$) s.t. $\bigwedge T_s \wedge \Psi \wedge \theta'$ ($\bigwedge T_s \wedge \neg\psi \wedge \theta''$) is large. Take $T_{s\frown\langle 0\rangle} = T_s \cup \{\psi, \theta'\}$ and $T_{s\frown\langle 1\rangle} = T_s \cup \{\neg\psi, \theta''\}$.

The proof of Theorem 1 is now complete.

Given a counterexample σ, call a formula $\phi(\mathbf{x})$ with free variables $\mathbf{x}$ ($\mathbf{x}$ a finite sequence) *large* (with respect to σ) if $\sigma \wedge \exists\mathbf{x}\phi(\mathbf{x})$ is large. Call $\phi(\mathbf{x})$ *minimal* (w.r.t. σ) if for all $\psi(\mathbf{x})$, $\phi \wedge \psi$ or $\phi \wedge \neg\psi$ is not large. Theorem 1 yields

COROLLARY 3. *Every large* (*w.r.t. a given counterexample* σ) *formula* $\phi(\mathbf{x})$ *can be strengthened to a minimal formula.*

PROOF. Notice that $\phi(\mathbf{x})$ is large iff $\sigma \wedge \phi(\mathbf{c})$ (a sentence in the larger language $L(\mathbf{c})$ with $\mathbf{c}$ a sequence of new constants) is a counterexample and that $\phi(\mathbf{x})$ is minimal w.r.t. σ iff $\sigma \wedge \phi(\mathbf{c})$ is a minimal counterexample. Now, the assertion follows from Theorem 1.

Now assume that σ is a minimal counterexample. For any fragment Δ containing σ, define $T_\Delta = \{\phi: \phi \in \Delta \text{ and } \phi \wedge \sigma \text{ is large}\}$. If Δ is countable then, by the minimality of σ, T_Δ is consistent (all but countably many of the countable models of σ are models of T_Δ) and Δ-complete. A formula $\psi(\mathbf{x}) \in \Delta$ is consistent with T_Δ iff it is large. This observation easily yields

LEMMA 4. *If* $\phi(\mathbf{x})$ *is minimal, then for all countable* Δ, *if* $\phi \in \Delta$ *then* ϕ *is complete* (*in* Δ) *with respect to* T_Δ (*cf. the definition on p.* 61 *of* [4]).

Call a fragment Δ *closed* if for every large $\phi(\mathbf{x}) \in \Delta$ there is a minimal formula $\phi'(\mathbf{x}) \in \Delta$ s.t. $\models \phi' \to \phi$. By Corollary 3, every countable fragment can be enlarged to a closed one. If Δ is countable and closed then, by Lemma 4 and the definition of T_Δ, there are no incompletable formulas $\psi(\mathbf{x})$ with respect to T_Δ. Hence, T_Δ has a prime model (cf. [4, pp. 61–64, especially Theorem 16]). Moreover, it is easily seen that if Δ is closed then a formula $\phi \in \Delta$ is complete w.r.t. T_Δ iff it is minimal. Hence, each finite sequence of elements in the prime model of T_Δ satisfies a minimal formula belonging to Δ.

As an application, we can now prove the following result (announced in [2]).

THEOREM 5. *If σ is a counterexample then it has an uncountable model N. Moreover, N can be so chosen as to satisfy only large sentences; thus, N is not $L_{\infty\omega}$ equivalent to any countable structure.*

PROOF. By Theorem 1, we may assume that σ is minimal. We define by induction an increasing chain of countable fragments Δ_α, and of countable structures M_α, $\alpha < \omega$, s.t.:

(i) $\sigma \in \Delta_\alpha$, and Δ_α is closed.

(ii) M_α is the prime model of T_{Δ_α}.

(iii) $M_\alpha \neq M_{\alpha+1}$, $M_\alpha \prec_{\Delta_\alpha} M_{\alpha+1}$ and $M_\lambda = \bigcup_{\alpha<\lambda} M_\alpha$ for a limit λ.

Once this is done, we shall take $N = \bigcup_{\alpha<\omega_1} M_\alpha$.

The inductive definition goes as follows:

Assume that Δ_α and M_α are defined.

Let $\Delta_{\alpha+1} \supset \Delta_\alpha$ be a closed fragment which contains a Scott sentence ϕ of M_α (cf. Chapter 2 in [4]). Let $M_{\alpha+1}$ be the prime model of $T_{\Delta_{\alpha+1}}$. As $M_{\alpha+1} \models T_{\Delta_\alpha}$ ($\subset T_{\Delta_{\alpha+1}}$) and M_α is a prime model of T_{Δ_α}, M_α can be embedded in $M_{\alpha+1}$. Thus, we may take $M_\alpha \prec_{\Delta_\alpha} M_{\alpha+1}$. As the Scott sentence ϕ of M_α is obviously not large, $\phi \notin T_{\Delta_{\alpha+1}}$, hence $M_{\alpha+1} \models \neg \phi$. This shows that $M_\alpha \not\cong M_{\alpha+1}$, hence, $M_\alpha \neq M_{\alpha+1}$.

For a limit λ, take $\Delta_\lambda = \bigcup_{\alpha<\lambda} \Delta_\alpha$ and $M_\lambda = \bigcup_{\alpha\in\lambda} M_\alpha$. Then Δ_λ is obviously closed and every finite sequence of elements of M_λ satisfies a minimal, hence by Lemma 4, complete (w.r.t. T_{Δ_λ}) formula. It follows (again by Theorem 16 in [4]) that M_λ is the prime model of T_{Δ_α}.

As said before, we take $N = \bigcup_{\alpha<\omega_1} M_\alpha$ to get an uncountable model of σ. If $N \models \phi$ then there is a closed and unbounded set $C \subset \omega_1$ s.t. $M_\alpha \models \phi$ for all $\alpha \in C$. By construction, $M_\alpha \not\cong M_\beta$ whenever $\alpha \neq \beta$; hence $\{M_\alpha: \alpha \in C\}$ is an uncountable collection of nonisomorphic countable models of ϕ, showing that ϕ is large. This completes the proof of Theorem 5.

An equivalent formulation of Theorem 5 says that, for a minimal counterexample σ, T_Δ is consistent even for uncountable fragments Δ, in particular for $\Delta = L_{\omega_1\omega}$.

The model N constructed in the proof of Theorem 5 has the further property that each finite sequence of it satisfies a minimal formula. This implies that N has Scott height ω_1 (the Scott height of a structure is the first α such that for all finite sequences $\mathbf{a}$ of elements of N, $N \models \forall \mathbf{x}(\phi_{\mathbf{a}}^{\alpha}(\mathbf{x}) \to \phi_{\mathbf{a}}^{\alpha+1}(\mathbf{x}))$ where $\phi_{\mathbf{a}}^{\alpha}$ is the $L_{\infty\omega}$ formula defined as in Chapter 2 of [4], without the restriction $\alpha < \omega_1$). Independently, Leo Harrington showed the stronger result that every counterexample has uncountable models of arbitrarily large Scott heights $\alpha < \omega_2$ (unpublished).

The second author of the present paper showed [5] that every counterexample has an uncountable model which is $L_{\infty\omega}$-equivalent to a countable one.

In a somewhat different context, we wish now to point out that the same tree argument that went into the proof of Theorem 1 can be used to replace Burgess' use in **[1]** of forcing in the deduction of his result on Σ_1^1 equivalence relations from Silver's theorem on Π_1^1 equivalence relation **[10]** (cf. also **[3]**).

By an *equivalence relation* E we mean one defined between countable L-structures which is weaker than isomorphism, i.e., if $M \cong N$ then $M(\mathrm{E})N$. Let L' be the (two sorted) language corresponding to the naturally defined disjoint sum $M \oplus N$ of any two L-structures M, N. The equivalence relation E is Borel (resp. Σ_1^1) if for some $\sigma \in L'_{\omega_1\omega}$ (resp. $\sigma(\mathbf{P}) \in L'(\mathbf{P})_{\omega_1\omega}$), $M(\mathrm{E})N$ iff $M \oplus N \models \sigma$ ($M \oplus N \models \exists \mathbf{P}\sigma(\mathbf{P})$). The following is an observation of Burgess:

Claim 6. Every Σ_1^1 equivalence relation is the intersection of $\aleph_1$ Borel equivalence relations.

Sketch of a model-theoretic proof. By Vaught **[11]** (see also **[6]**), there are $L'_{\omega_1\omega}$ sentences ϕ_α, $\alpha < \omega_1$, such that on countable structures $\exists \mathbf{P}\sigma(\mathbf{P})$ is equivalent to $\bigwedge_{\alpha<\omega_1}\phi_\alpha$. Set $M(\mathrm{E}_\alpha)N$ iff $M \oplus N \models \phi_\alpha$. The claim will follow if we show that E_α is an equivalence relation whenever α is an (admissible) ordinal such that $\sigma(\mathbf{P})$ is $\mathcal{A}$-finite for some admissible set with $\mathrm{ord}_{\mathcal{A}} = \alpha$. But this is easily established using the existence of $\Sigma_{\mathcal{A}}$-saturated models, as well as the fact that, in any $\Sigma_{\mathcal{A}}$-saturated structure, ϕ_α ($= \bigwedge_{\beta<\alpha}\phi_\beta$) is equivalent to $\exists \mathbf{P}\sigma(\mathbf{P})$ (cf. Corollary 7.3 and the proof of 8.1 in **[6]**; the notion of $\Sigma_{\mathcal{A}}$-saturated structure comes from **[9]**).

Using Claim 6 and a forcing argument, Burgess deduced from Silver's aforementioned theorem that any Σ_1^1 equivalence relation has $\leqslant \aleph_1$ or $2^{\aleph_0}$ equivalence classes. This result follows from the following.

THEOREM 7. *Assume that* $\aleph_0 < \kappa < 2^{\aleph_0}$. *If* E *is the intersection of* κ *many Borel equivalence relations then* E *has* $\leqslant \kappa$ *or* $2^{\aleph_0}$ *many equivalence classes.*

PROOF. Let $\mathrm{E} = \bigcap_{\alpha<\kappa}\mathrm{E}_\alpha$, each E_α a Borel equivalence relation; as is well known and easily seen, every equivalence class of any E_α is also Borel, i.e. definable by an $L_{\omega_1\omega}$ sentence.

Assume that E *has* $< 2^{\aleph_0}$ *equivalence classes*. Then the same is true for each E_α and by (a weakened version of) Silver's theorem, E_α has $\leqslant \aleph_0$ equivalence classes. For every equivalence class X of any E_α, $\alpha < \kappa$, select an $L_{\omega_1\omega}$ sentence defining it; collect all these sentences into a set Ψ. The properties of Ψ are summed up as follows:

(i) $|\Psi| \leqslant \kappa$;

(ii) each $\psi \in \Psi$ is E-invariant, i.e., $M \models \psi$ and $M(\mathrm{E})N$ imply that $N \models \psi$; and

(iii) Ψ distinguishes between the equivalence classes of E; i.e. if $\neg M(\mathrm{E})N$, then for some $\psi \in \Psi$, $M \models \psi$ and $N \models \neg\psi$.

Assume next, for proof by contradiction, that E *has* $> \kappa$ *equivalence classes*. For $\phi \in L_{\omega_1\omega}$, let $\mathcal{C}_\phi$ be the set of equivalence classes of E having a nonempty intersection with $\mathrm{Mod}(\phi)$. Call σ *large* iff $\mathcal{C}_\sigma$ has power $> \kappa$. A simple counting argument shows:

Claim 8. For every large σ there is $\psi \in \Psi$ s.t. both $\sigma \wedge \psi$ and $\sigma \wedge \neg \psi$ are large.

PROOF. Let Γ be the set of those sentences γ which belong to Ψ or are negations of members of Ψ s.t. $\sigma \wedge \gamma$ is not large. Then

$$\mathcal{C}_\sigma = \bigcup \{\mathcal{C}_{\sigma\wedge\gamma} : \gamma \in \Gamma\} \cup \bigcap \{\mathcal{C}_{\sigma\wedge\neg\gamma} : \gamma \in \Gamma\}.$$

If the claim is false, then the second term of the union is easily seen, by (iii) above, to contain just one element. Thus, by (i), $|\mathcal{C}_\sigma| \leqslant \kappa$, contradicting the largeness of σ.

Now, using Claim 8, one constructs precisely as in the proof of Theorem 1, a tree T_s, $s \in {}^{<\omega}2$ s.t. for all s:

(a) T_s is finite, $\sigma \in T_s$ and $\bigwedge T_s$ is large;

(b) there is $\psi \in \Psi$ s.t. $\psi \in T_{s^\wedge\langle 0\rangle}$ and $\neg\psi \in T_{s^\wedge\langle 1\rangle}$, and such that for all $\eta \in {}^\omega 2$, $T_\eta = \bigcup_{n<\omega} T_{\eta\restriction n}$ has a model M_η. By (ii) and (b) above, $\neg M_\eta(\mathrm{E})M_{\eta'}$ whenever $\eta \neq \eta'$. Thus E has $2^{\aleph_0}$ equivalence classes, a contradiction.

We conclude by indicating a most natural example of a relation which is the intersection of $\aleph_1$ Borel equivalence relations. If K is the class of countable members of a $PC_{\omega_1\omega}$ class, define: $M(\mathrm{E})N$ iff $M, N \notin \mathrm{K}$ or $M \cong N$. Again by [11], $\mathrm{K} = \bigcap_{\alpha<\omega_1} \mathrm{K}_\alpha$ where each K_α is Borel. Define: $M(\mathrm{E}_\alpha)N$ iff $M, N \notin \mathrm{K}_\alpha$ or $M \equiv_\alpha N$ (where $\equiv_\alpha$ means equivalence w.r.t. $L_{\omega_1\omega}$ sentences with quantifier rank $\leqslant \alpha$). Obviously, $\mathrm{E} = \bigcap_{\alpha<\omega_1} \mathrm{E}_\alpha$ and each E_α is Borel. Thus, Theorem 7 implies Morley's result [7] that any $PC_{\omega_1\omega}$ class has $\leqslant \aleph_1$ or $2^{\aleph_0}$ countable nonisomorphic models. (Actually, Morley's proof contains an argument showing that the particular E_α defined above satisfies Silver's theorem.)

ADDED MAY 30, 1977. In [12], John Burgess proves a theorem which is more general than our Theorem 7. Burgess' proof is forcing-free as well. We are indebted to the referee for this information.

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