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NEAR COMPACTNESS AND SEPARABILITY OF SYMMETRIZABLE SPACES

R. M. STEPHENSON, JR.

ABSTRACT. Although every feebly compact, Baire, semimetrizable space is separable, we prove here that for every infinite cardinal number n there exists a feebly compact, Baire, symmetrizable Hausdorff space which has no dense subset of cardinality less than n.

For a topological space X, a mapping d: $X \times X \to [0, \infty)$ is said to be a *symmetric* provided that: (i) for all $x, y \in X$, $d(x,y) = d(y, x)$, and $d(x,y) = 0$ if and only if $x = y$; and (ii) for any subset V of X, V is open if and only if for each point $v \in V$ there exists $e > 0$ with $B(v, e) \equiv \{x \in X: d(x, v) < e\} \subset V$. If, in addition, each $B(x, e)$, $x \in X$, $e > 0$, is a neighborhood of x, then d is called a *semimetric*. A space X which has a symmetric (semimetric) is said to be *symmetrizable* (semimetrizable).

A. V. Arhangel'skiĭ [**A**, p. 126] proved that every countably compact symmetrizable Hausdorff space is metrizable, and in [**S1**] and [**S2**] properties of symmetrizable feebly compact spaces were studied (recall that a space X is said to be *feebly compact* if every locally finite family of open subsets of X is finite). Of particular interest there was the question: *Is every feebly compact symmetrizable space separable*? Proofs were given in [**S1**] that every feebly compact symmetrizable space having a dense set of isolated points is separable, and in [**S2**, Theorem 10] that every feebly compact, Baire, semimetrizable space is separable. The latter extended Reed's theorem [**R**] that every Moore-closed space is separable, for a Moore-closed space is regular and feebly compact [**G**], and a regular, feebly compact space is Baire [**M**].

In this paper, a modification of a very nice technique developed in [**DGN**, Example 3.1] is used to settle the question in the negative, and we obtain the following surprising result.

THEOREM. *Let n be an infinite cardinal number. Then there exists a Baire, feebly compact, symmetrizable Hausdorff space X such that no dense subset of X has cardinality less than $m = n^{\aleph_0}$.*

PROOF. Let Y be a metrizable Baire space such that $|V| = m$ for every nonempty open subset V of Y, and $|D| = m$ for any dense subset D of Y. Let d' be a metric for Y, $\mathfrak{B}$ a base for Y with $|\mathfrak{B}| = m$, and **C** be the family of all

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countably infinite, pairwise disjoint, locally finite families of nonempty members of $\mathfrak{B}$.

List the members of the collection $\mathbf{C}$ as $\mathbf{C} = \{\mathcal{C}_k\colon k < m\}$ and list the members of each $\mathcal{C}_k$ in a 1-1 manner as $\mathcal{C}_k = \{C_{k,j}\colon j \in \mathbf{N}\}$. Since each $|C_{k,j}| = m$, one can by transfinite induction select points $s_{k,j} \in C_{k,j}$, where $k < m$ and $j \in \mathbf{N}$, so that whenever $i, k \in m$ and $i \neq k$, then

$$\{s_{i,j}\colon j \in \mathbf{N}\} \cap \{s_{k,j}\colon j \in \mathbf{N}\} = \varnothing.$$

Let $X = Y \cup m$ and extend d' to a symmetric d on X by the rule

$$d(x,y) = d(y,x) = \begin{cases} 0 & \text{if } x = y; \\ d'(x,y) & \text{if } x, y \in X; \\ 1/j & \text{if } x = k \text{ and } y = s_{k,j}; \text{ and} \\ 1 & \text{otherwise.} \end{cases}$$

Next, let X have the topology induced on it by d.

Before verifying that X is Hausdorff, observe that for each point $y \in Y$, one has $y = s_{k,j}$ for at most one pair k, j, so for each $y \in Y$ there exists $e(y) > 0$ with $B(y, e(y)) \subset Y$. Thus $\{B(y, e)\colon 0 < e \leqslant e(y)\}$ is a fundamental system of open neighborhoods of y in X. For a point $k < m$, a fundamental system of open neighborhoods is the family of all sets having the form

$$\{k\} \cup \left(\bigcup \{B(s_{k,j}, f_j)\colon 0 < f_j \leqslant e(s_{k,j}), t \leqslant j\}\right),$$

where $t \in \mathbf{N}$ and f is a sequence of real numbers, the jth term of which is f_j.

Consider distinct points x and y in X. If both are in the metrizable open subset Y, then disjoint neighborhoods can certainly be found. Suppose $x = k < m$ and $y \in Y$. For some $t \in \mathbf{N}$, $\{y\}$ and $\{s_{k,j}\colon j \geqslant t\}$ are disjoint closed subsets of Y (since $\mathcal{C}_k$ is locally finite in Y and pairwise disjoint), so there exist disjoint open subsets U and V of Y with $y \in U$ and $\{s_{k,j}\colon j \geqslant t\} \subset V$. Thus, U and $V \cup \{k\}$ are disjoint neighborhoods of y and x. If $x = k$ and $y = r$ with $k, r < m$, then one can (again) appeal to the normality of Y and topology on X to find disjoint open sets containing $\{k\} \cup \{s_{k,j}\colon j \in \mathbf{N}\}$ and $\{r\} \cup \{s_{r,j}\colon j \in \mathbf{N}\}$.

Because Y is a dense, Baire subspace of X, the space X must also be Baire. Since Y is an open subspace having no dense subset of cardinality less than m, then X has no dense subset of cardinality less than m.

Finally, suppose that $\mathcal{V}$ is an infinite family of open subsets of X. We will prove that $\mathcal{V}$ fails to be locally finite.

Suppose, on the contrary, that $\mathcal{V}$ is locally finite. Since Y is dense in X, one can find a countably infinite pairwise disjoint family $\mathcal{W}$ of members of $\mathfrak{B}$ and a 1-1 mapping $f\colon \mathcal{W} \to \mathcal{V}$ such that for each $W \in \mathcal{W}$, one has $W \subset f(W)$. Evidently any point at which $\mathcal{W}$ fails to be locally finite must also be a point at which $\mathcal{V}$ fails to be locally finite. Thus $\mathcal{W}$ is locally finite with respect to Y, and hence $\mathcal{W} = \mathcal{C}_k$ for some $k < m$. But clearly $\mathcal{C}_k$ fails to be locally finite at the point k, so we have a contradiction.

REMARKS. (i) I do not know if every regular, feebly compact symmetrizable space is separable. Since a G_δ-point in a regular, feebly compact space must have a countable neighborhood base (by an observation of I. Glicksberg), and since a first countable symmetrizable Hausdorff space is semimetrizable, any example of a regular, feebly compact, symmetrizable space that is not separable would also provide a negative answer to the still open question (see **[DGN]**) as to whether or not every point of a regular symmetrizable space must be a G_δ.

(ii) In the construction above, if Y is chosen so that no compact subset of Y has nonempty interior, then arguments similar to ones given in **[DGN]** show that X has a closed subset, namely m, which fails to be a G_δ-set (because then if $\mathcal{V}$ is a countable family of open sets containing m, the family $\mathcal{W} = \{V \cap Y: V \in \mathcal{V}\}$ consists of dense open subsets of Y, and so $\varnothing \neq \bigcap \mathcal{W} \subset Y$ and $\bigcap \mathcal{V} \neq m$).

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DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, UNIVERSITY OF SOUTH CAROLINA, COLUMBIA, SOUTH CAROLINA 29208