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A CHARACTERIZATION OF THE PEDERSEN IDEAL OF $C_0(T, B_0(H))$ AND A COUNTEREXAMPLE

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ABSTRACT. Let T be a locally compact Hausdorff space, H a complex Hilbert space, and A the C^*-algebra $C_0(T, B_0(H))$. Let A_0 be the Pedersen ideal of A and J_A the two-sided ideal of A consisting of all x having compact support, for which $\sup\{\dim x(t): t \in T\} < \infty$. It is known that $A_0 \subseteq J_A$, and equality has been conjectured by Pedersen. We give a new characterization of A_0 which enables us to show that the conjecture is false.

1. Introduction. Let A be a C^*-algebra with continuous trace, $\hat{A}$ the spectrum of A, and J_A the set of all x in A such that $\sup\{\dim \pi(x): \pi \in \hat{A}\} < \infty$ and $\pi(x) = 0$ for π outside some compact subset of $\hat{A}$. In [**2**, 4.7.24, p. 100] Dixmier asked whether or not J_A is the minimal dense two-sided ideal of A. Pedersen and Petersen answered this question negatively in [**9**, Proposition 3.6, p. 202]. By using homogeneous algebras whose corresponding fibre bundles have sufficiently many twists, Pedersen and Petersen were able to construct an example of a C^*-algebra A with continuous trace for which J_A is not the minimal dense two-sided ideal of A. In [**8**, p. 13] Pedersen did conjecture, however, that when $A = C_0(T, B_0(H))$, then J_A is the minimal dense hereditary two-sided ideal of A, or equivalently, the minimal dense two-sided ideal (see [**4**, 2, p. 168]). Here T is a locally compact Hausdorff space and $B_0(H)$ is the C^*-algebra of compact operators on some Hilbert space H. The minimal dense hereditary (order related) two-sided ideal of a C^*-algebra is commonly referred to as Pedersen's ideal; this ideal was shown to exist in every C^*-algebra by Pedersen [**6**], [**8**].

In §2 of this note we give a new characterization of Pedersen's ideal of $C_0(T, B_0(H))$. Consequently, in §3 we are able to construct an example that shows Pedersen's conjecture is false. For basic concepts and definitions we refer the reader to [**2**], [**6**], [**8**].

2. Pedersen's ideal of $C_0(T, B_0(H))$. Let T be a locally compact Hausdorff space and H a Hilbert space. Let $\mathfrak{N} = \mathfrak{N}(T)$ denote the set of all ordered triples $n = (U, \alpha, e)$ that satisfy the following:

(i) U is an open subset of T;

(ii) α is a nonnegative continuous function defined on T which has compact support and for which $\{t \in T: \alpha(t) > 0\} \subseteq U$;

(iii) e is continuous mapping of U into H such that $\|e(t)\| = 1$ for all $t \in U$

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(the topology for H is the norm topology).

For each $n = (U, \alpha, e)$ define the map z_n: $T \to B_0(H)$ by

$$z_n(t) = \begin{cases} \alpha(t)P_e(t), & t \in U, \\ 0 & \text{otherwise,} \end{cases}$$

where $P_e(t)$ denotes the projection of H onto H_t, the subspace of H generated by $e(t)$, and $B_0(H)$ denotes the C^*-algebra of all compact operators on H. Let $C_0(T, B_0(H))$ denote the C^*-algebra of all continuous maps x: $T \to B_0(H)$ such that the real map $t \to \|x(t)\|$ vanishes at infinity. Here the topology for $B_0(H)$ is the norm topology. Finally, let A denote the C^*-algebra $C_0(T, B_0(H))$ and A_0 its Pedersen ideal.

2.1. Lemma. *Let* $D = \{z_n: n \in \mathfrak{N}\}$. *Then the following statements hold*: (a) $D \subseteq A^+$; (b) $D = \{z^{1/2}: z \in D\}$; (c) $xDx^* \subseteq D$, *for all* $x \in A$; (d) *if* $0 \leqslant x \leqslant z$, *where* $x \in A$ *and* $z \in D$, *then* $x \in D$; (e) *if* $u \in A$ *and* $u^*u \in D$, *then* $uu^* \in D$.

Proof. Clearly, (a), (b), and (d) hold. Now let $x \in A$ and $n = (U, \alpha, e) \in \mathfrak{N}$. It is clear that the map $t \to x(t)[e(t)]$ is continuous on U; hence, $V = \{t \in U: 0 < \|x(t)[e(t)]\|\}$ is an open subset of T. Define

$$f(t) = (1/\|x(t)[e(t)]\|)(x(t)[e(t)])$$

for each $t \in V$. Set

$$\beta(t)\begin{cases} \|x(t)[e(t)]\|^2\alpha(t), & t \in V, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $\beta(t)$ is a nonnegative continuous function defined on T with compact support and $\{t \in T: \beta(t) > 0\} \subseteq V$. Now set $m = (V, \beta, f)$, which certainly belongs to $\mathfrak{N}$. It is straightforward to show that $xz_nx^* = z_m$. Hence (c) holds. Finally, suppose $u \in A$ and $u^*u = z \in D$. Then $(uu^*)^2 = uzu^* \in D$ by (c), hence $uu^* \in D$ by (b). So (e) holds and our proof is complete.

2.2. Theorem. *Let*

$$I = \left\{ \sum_{n \in \mathfrak{F}} z_n: \mathfrak{F} \subseteq \mathfrak{N}, \mathfrak{F} \textit{ finite} \right\}.$$

Then I is the minimal-dense, invariant order ideal (*face*) *of* A^+, *that is*, span I *is the Pedersen ideal of* A.

Proof. Let $x \in A^+$ be so that $x \leqslant \sum_{i=1}^p z_{n_i}$ for some finite subset $n_1, n_2, \ldots, n_p$ of $\mathfrak{N}$. By the Riesz decomposition property [7, Corollary 2, p. 267], there are elements $u_1, u_2, \ldots, u_p$ in A so that $x = u_1u_1^* + \cdots + u_pu_p^*$ and $u_i^*u_i \leqslant z_{n_i}$, $i = 1, 2, \ldots, p$. It follows from 2.1(d), (e) that $x \in I$, so I is an order ideal (face) of A^+. Furthermore, by 2.1(c), I is an invariant order ideal of A^+ and by [2, 10.5.3, p. 199], span I is dense in A. Thus I is a dense invariant order ideal, so $A_0^+ \subseteq I$. To show $I = A_0^+$, it suffices to observe $D \subseteq A_0$. Let $n = (U, \alpha, e) \in \mathfrak{N}$ and choose $h_0 \in H$ so that $\|h_0\| = 1$.

Without loss of generality we may assume $\|\alpha^{1/2}\|_\infty < \frac{1}{2}$. Now set

$$f(t) = \begin{cases} \dfrac{h_0 - \alpha^{1/2}(t)e(t)}{\|h_0 - \alpha^{1/2}(t)e(t)\|}, & t \in U, \\ h_0, \quad t \notin U, \end{cases}$$

and

$$g(t) = \begin{cases} \dfrac{h_0 + \alpha^{1/2}(t)e(t)}{\|h_0 + \alpha^{1/2}(t)e(t)\|}, & t \in U, \\ h_0, \quad t \notin U. \end{cases}$$

Clearly, the maps $t \to f(t)$ and $t \to g(t)$ are continuous on all of T. From [**8**, p. 8], we see that, for each $\beta \in C_{00}(T)^+$, βP_f and βP_g belong to A_0^+. So choose $\beta \in C_{00}(T)^+$ with $\beta(t) = 1$, $t \in \operatorname{supp} \alpha$, and $\|\beta\|_\infty \leq 1$. Now let $h \in H$ and let $t \in T$ be such that $\alpha(t) > 0$. Note

$$\begin{aligned} \langle \alpha(t)P_e(t)[h], h\rangle &= \alpha(t)|\langle h, e(t)\rangle|^2 \\ &\leq 2\alpha(t)|\langle e(t), h\rangle|^2 + 2|\langle h, h_0\rangle|^2 \\ &= |\langle h, h_0\rangle|^2 - 2\mathrm{Re}\, \alpha^{1/2}(t)\langle h, e(t)\rangle\langle h_0, h\rangle + \alpha(t)|\langle e(t), h\rangle|^2 \\ &\quad + |\langle h, h_0\rangle|^2 + 2\mathrm{Re}\, \alpha^{1/2}(t)\langle h, e(t)\rangle\langle h_0, h\rangle + \alpha(t)|\langle e(t), h\rangle|^2 \\ &= |\langle h, h_0 - \alpha^{1/2}(t)e(t)\rangle|^2 + |\langle h, h_0 + \alpha^{1/2}(t)e(t)\rangle|^2 \\ &= \|h_0 - \alpha^{1/2}(t)e(t)\|^2|\langle h, f(t)\rangle|^2 \\ &\quad + \|h_0 + \alpha^{1/2}(t)e(t)\|^2\, |\langle h, g(t)\rangle|^2 \\ &\leq 4|\langle h, f(t)\rangle|^2 + 4|\langle h, g(t)\rangle|^2 \\ &= 4\langle P_f(t)[h], h\rangle + 4\langle P_g(t)[h], h\rangle \\ &= 4\langle \beta(t)P_f(t)[h], h\rangle + 4\langle \beta(t)P_g(t)[h], h\rangle. \end{aligned}$$

Thus $z_n \leq 4\beta P_f + 4\beta P_g$. Since A_0^+ is an order ideal (face) of A^+, $z_n \in A_0^+$. So $D \subseteq A_0$ and our proof is complete.

3. Examples. We now detail the construction of a compact Hausdorff space T and an element x of the C^*-algebra $A = C(T, B_0(H))$ which does not belong to the Pedersen ideal of A, even though each $x(t)$ is a positive operator on H having dimension at most 1. The Hilbert space H is required to be infinite dimensional.

The building blocks for the space T are the complex projective spaces P^m, which are defined as follows: P^m is the set of all 1-dimensional subspaces of $\mathbf{C}^{m+1}$, topologized as a quotient space of $\mathbf{C}^{m+1} \sim \{0\}$. The space P^m is a compact metric space. By identifying $\mathbf{C}^{m+1}$ with a fixed subspace of H, we can view a point π of P^m as a 1-dimensional subspace of H. To this subspace π we assign the projection operator $x_m(\pi)$ which projects H onto π. Since P_h

(the projection of H onto the span of h) is continuous in h, for $h \in H \sim \{0\}$, and since $x_m(\pi) = P_h$ whenever $h \in \pi \sim \{0\}$, it follows that x_m is a continuous function from P^m to $B_0(H)$. Moreover, x_m belongs to the Pedersen ideal of the C^*-algebra $C(P^m, B_0(H))$, because x_m is positive and $x_m^2 = x_m$. The characterization of the Pedersen ideal given in the previous section applies to x_m with the result that for some finite sequence $n(1), \ldots, n(k)$ in $\mathfrak{N}(P^m)$,

$$x_m = \sum_{i=1}^{k} z_{n(i)}. \tag{1}$$

Let $\gamma(x_m)$ denote the smallest integer k for which such a sequence $n(1), \ldots, n(k)$ exists. We will now prove that $\gamma(x_m) \geqslant m + 1$. This is the key to our example, and it is here that global topological properties of P^m enter.

Let γ_1^{m+1} be the canonical complex line bundle over P^m. The total space E of γ_1^{m+1} consists of all pairs (π, v) such that $\pi \in P^m$ and $v \in \pi$. The projection $p\colon E \to P^m$ is defined by $p(\pi, v) = \pi$. Suppose now that (1) holds with $n(i) = (U_i, \alpha_i, e_i)$, and let V_i be the open subset of U_i on which α_i is strictly positive. The sets $V_1, V_2, \ldots, V_k$ cover P^m because x_m is never zero. Since x_m has rank 1 everywhere, it follows from (1) that if $\pi \in V_i$, then $x_m(\pi) = P_{e_i}(\pi)$; or what amounts to the same thing, $e_i(\pi) \in \pi$. We conclude that $(\pi, e_i(\pi)) \in E$ and $p(\pi, e_i(\pi)) = \pi$ whenever $\pi \in V_i$, which is precisely the statement that the bundle γ_1^{m+1} admits a cross-section over V_i. Since this cross-section is never zero, γ_1^{m+1} is trivial over V_i [3, Exercise 1, p. 37]. Because each restriction $\gamma_1^{m+1}|V_i$ is trivial ($i = 1, 2, \ldots, k$) there is a mapping $f\colon P^m \to P^{k-1}$ such that $\gamma_1^{m+1} \cong f^*\gamma_1^k$, where $f^*\gamma_1^k$ is the induced bundle [3, Proposition 5.8, p. 31, and the proof of Theorem 5.5, p. 30]. The Chern class $c_1(\gamma_1^k)$ generates the integral cohomology ring $H^*(P^{k-1}, Z)$ and is carried by the induced cohomology homomorphism onto the Chern class $c_1(f^*\gamma_1^k)$ [3, pp. 232–233], [5, p. 160]:

$$f^*c_1(\gamma_1^k) = c_1(f^*\gamma_1^k) = c_1(\gamma_1^{m+1}). \tag{2}$$

We can conclude from (2) that $k > m$ because the kth power of $c_1(\gamma_1^k)$ is zero. This completes the proof that $\gamma(x_m) \geqslant m + 1$. (We are grateful to the referee for suggesting this proof.) We summarize our results in a theorem.

3.1. THEOREM. *Assume that H is infinite dimensional. For each positive integer m, the C^*-algebra $C(P^m, B_0(H))$ contains an element x_m such that $x_m(\pi)$ is a 1-dimensional projection for each $\pi \in P^m$, and for which $\gamma(x_m) \geqslant m + 1$.*

Returning to the construction of our example, define T to be the one-point compactification of the disjoint union of the P^m, $m = 1, 2, \ldots$:

$$T = \{\omega\} \cup \bigcup_{m=1}^{\infty} P^m.$$

Define an element x of the C^*-algebra $A = C(T, B_0(H))$ by the formula

$$x(t) = \begin{cases} m^{-1}x_m(t) & \text{if } t \in P^m, \\ 0 & \text{if } t = \omega. \end{cases}$$

For each $t \in T$, $x(t)$ is positive and has dimension at most 1. However, x cannot belong to the Pedersen ideal of A because if it does, there must exist a finite sequence $n(1), \ldots, n(k)$ in $\mathfrak{N}(T)$ such that

$$x = \sum_{i=1}^{k} z_{n(i)}, \tag{3}$$

and by choosing an integer $m \geqslant k$ and restricting the terms of (3) to P^m, we obtain a sum of form (1) with $k \leqslant m$, contrary to Theorem 3.1. (When restricting the terms of (3) to P^m we must also restrict the members of each triple $n(i)$ to P^m.) We state these results in the form of a theorem.

3.2. THEOREM. *Assume that H is an infinite dimensional Hilbert space. There exists a compact metric space T such that $C(T, B_0(H))$ contains a positive x having dimension everywhere less than or equal to* 1, *which does not belong to the Pedersen ideal of $C(T, B_0(H))$.*

It is worth pointing out that this example shows us the role of the mappings e_i in our characterization of the Pedersen ideal. The example x constructed above can be written in the form

$$x(t) = \begin{cases} \alpha(t)P(t), & t \neq \omega, \\ 0, & t = \omega, \end{cases}$$

where P is a continuous projection valued map on $T \sim \{\omega\}$, and where $\alpha \in C(T)$.

REFERENCES

1. J. Dixmier, *Les algèbres, d'opérateurs dans l'espace Hilbertien*, 2[e] éd., Gauthier-Villars, Paris, 1967.

2. ______, *Les C*-algèbres et leur représentations*, Gauthier-Villars, Paris, 1964.

3. D. Husemoller, *Fibre bundles*, McGraw-Hill, New York, 1966.

4. K. Laursen and A. Sinclair, *Lifting matrix units in C*-algebras*. II, Math. Scand. **37** (1975), 167–172.

5. J. Milnor and J. Stasheff, *Characteristic classes*, Ann. of Math. Studies, no. 76, Princeton Univ. Press, Princeton, N.J., 1974.

6. G. Pedersen, *Measure theory for C*-algebras*, Math. Scand. **19** (1966), 131–145.

7. ______, *A decomposition theorem for C*-algebras*, Math. Scand. **22** (1968), 266–268.

8. ______, *C*-integrals, an approach to non-commutative measure theory*, Copenhagen Univ., Copenhagen.

9. G. Pedersen and N. Petersen, *Ideals in a C*-algebra*, Math. Scand. **27** (1970), 193–204.

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