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FREE LIE ALGEBRAS AS MODULES OVER THEIR ENVELOPING ALGEBRAS

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ABSTRACT. In this paper we determine the linear relations that exist between the free generators of a free Lie algebra L when it is viewed as a module over its enveloping algebra via the adjoint representation. As an application, the annihilator of a homogeneous element of L is determined.

1. Statement of results. Let K be a commutative associative ring with unit, let L be a free Lie algebra over K (cf. [**1**]), and let U be the enveloping algebra of L. We identify L with its image under the canonical injection of L into U. We are interested in the structure of L as a left U-module via the adjoint representation ad: $U \to \mathrm{End}_K(L)$.

Let $x_1, \ldots, x_n$ be arbitrary nonzero elements of L. Let $e_1, \ldots, e_n$ be the usual basis of the (left) U-module U^n:

$$e_i = (d_{i1}, \ldots, d_{in})$$

with $d_{ik} = 1$ for $k = i$ and zero otherwise. For $1 \leqslant i, j \leqslant n$, $u, v \in U$, define $e_i(u)$, $e_{ij}(u, v) \in U^n$ by

$$e_i(u) = (\mathrm{ad}(u)x_i)ue_i,$$

$$e_{ij}(u, v) = (\mathrm{ad}(v)x_j)ue_i + (\mathrm{ad}(u)x_i)ve_j,$$

and let E be the U-submodule of U^n generated by the elements $e_i(u)$, $e_{ij}(u, v)$ with $1 \leqslant i, j \leqslant n$, $u, v \in U$. If $x_1, \ldots, x_n$ are homogeneous for some grading of the Lie algebra L, then the elements u, v above can be taken to be homogeneous for the natural grading of U induced by the grading of L. Hence, in this case, E is a homogeneous submodule for the grading of U^n defined by saying that $(u_1, \ldots, u_n) \in U^n$ is homogeneous of degree k if and only if $u_1, \ldots, u_n \in U$ are homogeneous of degree k. If $(u_1, \ldots, u_n) \in E$, then we always have the relation

$$\mathrm{ad}(u_1)x_1 + \cdots + \mathrm{ad}(u_n)x_n = 0.$$

THEOREM 1. *If $x_1, \ldots, x_n$ is a free generating system for L, then*

$$\mathrm{ad}(u_1)x_1 + \cdots + \mathrm{ad}(u_n)x_n = 0 \quad \textit{iff } (u_1, \ldots, u_n) \in E.$$

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The proof of this theorem (for which we are indebted to the referee) is a minor adaptation of an argument in [**2**].

Now let I be the ideal of L generated by $x_1, \dots, x_n$ and let W be the enveloping algebra of L/I. Consider the following conditions on $x_1, \dots, x_n$:

(1) *The elements* $x_1, \dots, x_n$ *are homogeneous for some grading of* L;

(2) *The ideal* I *is a free Lie algebra*;

(3) *The quotient* L/I *is a free* K*-module*;

(4) *The quotient* $I/[I, I]$ *is a free* W*-module of rank* n *with basis* $x_1, \dots, x_n$ (mod $[I, I]$).

These conditions are satisfied if $x_1, \dots, x_n$ is part of a free generating system for L (cf. [**1**, §2, Proposition 10]) or if $n = 1$, K is a field, and x_1 is homogeneous and nonzero (cf. [**3**]).

THEOREM 2. *If conditions* (1), (2), (3), (4) *hold, then the conclusion of Theorem* 1 *remains valid.*

COROLLARY 1. *If* $x_1, \dots, x_n$ *is part of a free generating system for* L, *then*

$$\operatorname{ad}(u_1)x_1 + \cdots + \operatorname{ad}(u_n)x_n = 0 \quad \textit{iff } (u_1, \dots, u_n) \in E.$$

COROLLARY 2. *If* x *is a nonzero homogeneous element of* L, *and if* K *is a field, then* $\operatorname{ad}(u)x = 0$ *if and only if*

$$u = \sum_{v \in U} w_v(\operatorname{ad}(v)x)v \qquad (w_v \in U).$$

In [**4**] we use Corollary 2 in an essential way to determine the Lie algebra associated to the lower central series of the group $\langle x, y \colon x^p = 1 \rangle$ (p a prime). We do not know whether the homogeneity condition can be dropped in Corollary 2.

As was pointed out to us by the referee, Corollary 2 includes as a special case the known result (cf. [**5**, Theorem 5.10]) that two homogeneous elements of a free Lie algebra over a field K commute if and only if they are linearly dependent. However, the result in [**5**] is more general since there it is not assumed that K is a field.

2. Proof of Theorem 1. Complete $x_1, \dots, x_n$ to a Hall basis H of L (cf. [**1**]). The elements of H are homogeneous (for the natural grading of L), form an ordered set, and are defined inductively as follows:

(i) The elements $u \in H$ of degree $d(u) = 1$ are $x_1, \dots, x_n$.

(ii) The elements $u \in H$ of a given degree $\geqslant 1$ are ordered in an arbitrary manner. If $u, v \in H$, then $u < v$ if $d(u) < d(v)$.

(iii) If $u, v \in H$ with $u < v$, then $[u, v] \in H$ if $d(v) = 1$ or if $v = [v_1, v_2]$ with $v_1, v_2 \in H$ and $v_2 > v_1 \leqslant u$.

Every element of H can be uniquely written in the form $\operatorname{ad}(u_k \dots u_1)x_j$ where (a) $k \geqslant 0$; (b) $u_i \in H$ for $1 \leqslant i \leqslant k$; (c) $u_1 \leqslant u_2 \leqslant \cdots \leqslant u_k$; (d) $\operatorname{ad}(u_{k-i} \dots u_1)x_j > u_{k-i+1}$ for $1 \leqslant i \leqslant k$. Conversely, such elements are elements of H. We call an element $u_k \dots u_1$ *normed of type* j if (a), (b), (c), (d) are satisfied.

Let N be the K-submodule of U^n spanned by the elements of the form we_i, where w is normed of type i. The mapping $f\colon U^n \to L$ defined by

$$f(u_1, \ldots, u_n) = \operatorname{ad}(u_1)x_1 + \cdots + \operatorname{ad}(u_n)x_n$$

is a U-module homomorphism of U^n onto L whose restriction to N is bijective. To prove the theorem it suffices to show that $U^n = N + E$ since $E \subseteq \operatorname{Ker}(f)$. But this would follow if we could show that $vN \subseteq N + E$ for any $v \in H$. Let $v = \operatorname{ad}(v_1 \ldots v_l)x_i$, where $v_1 \ldots v_l$ is normed of type i, let $u_1 \ldots u_k$ be normed of type j, and let $u = \operatorname{ad}(u_1 \ldots u_k)x_j$. We want to show that $vu_1 \ldots u_k e_j \in N + E$. Let $m = d(u) + d(v)$.

If $m = 2$, we have $k = 0$ and $v = x_i$. If $x_i = x_j$, we have $ve_j \in E$. If $x_i < x_j$, then v is normed of type j and $ve_j \in N$. If $x_j < x_i$, then

$$ve_j = x_i e_j \equiv -x_j e_i \pmod{E}$$

and, since x_j is normed of type i, we have $ve_j \in N + E$. Hence the result holds if $m = 2$.

We proceed by induction on m, assuming that $m > 2$ and that the result holds for all pairs (u', v') with $d(u') + d(v') < m$. We have

$$\begin{aligned} vu_1 \ldots u_k e_j &= (\operatorname{ad}(v_1 \ldots v_l)x_i)u_1 \ldots u_k e_j \\ &\equiv -(\operatorname{ad}(u_1 \ldots u_k)x_j)v_1 \ldots v_l e_i \pmod{E} \\ &= -uv_1 \ldots v_l e_i. \end{aligned}$$

Also $uu_1 \ldots u_k e_j = (\operatorname{ad}(u_1 \ldots u_k)x_j)u_1 \ldots u_k e_j \in E$. Hence, without loss of generality, we may assume that $v < u$. Hence $\min(d(u), d(v)) = d(v)$. As $m > 2$ we have $d(u) > 1$, and so $k > 0$.

(a) If $v \geqslant u_1$, then (as $v < u$) $vu_1 \ldots u_k$ is normed of type j and so $vu_1 \ldots u_k e_j \in N$.

(b) If $\min(d(u), d(v)) > m/3$, then $d(u) = m - d(v) < 2m/3$. As $u_1 \ldots u_k$ is normed of type j we have $u_1 < \operatorname{ad}(u_2 \ldots u_k)x_j$ and so $d(u_1) \leqslant d(u)/2 < m/3$. Thus $d(u_1) < d(v)$ and hence $u_1 < v$. Then by (a) the result holds.

(c) Proceeding by downward induction on $\min(u, v)$, we assume that the result holds for all pairs (w, z) with $z \in H$, $w = \operatorname{ad}(w_1 \ldots w_r)x_s$, $w_1 \ldots w_r$ normed of type s, $d(w) + d(z) = m$, and $\min(w, z) > \min(u, v)$. In view of (a), we may assume $u_1 > v$. Now

$$\begin{aligned} vu_1 \ldots u_k e_j &= (\operatorname{ad}(v_1 \ldots v_l)x_i)u_1 \ldots u_k e_j \\ &\equiv -(\operatorname{ad}(u_1 \ldots u_k)x_j)v_1 \ldots v_l e_i \pmod{E} \\ &= -u_1\{(\operatorname{ad}(u_2 \ldots u_k)x_j)v_1 \ldots v_l e_i\} \\ &\qquad + (\operatorname{ad}(u_2 \ldots u_k)x_j)\{u_1 v_1 \ldots v_l e_i\}. \end{aligned}$$

The expressions in braces both have degree $< m$. Thus by the induction assumption they are congruent modulo E to elements of N of the same degree. Now $u_1 > v$, $\operatorname{ad}(u_2 \ldots u_k)x_j > u_1 > v$, $d((\operatorname{ad}(u_2 \ldots u_k)x_j)v_1 \ldots v_l) \geqslant d(v)$, and $d(u_1 v_1 \ldots v_l) \geqslant d(v)$. Consequently $vu_1 \ldots u_k e_j$ is a linear combination of terms of the form $zw_1 \dots w_r e_s$ with $z \in H$, $w_1 \dots w_r$ normed of type s, $z > v$, and $w = \mathrm{ad}(w_1 \dots w_r)x_s > v$. By induction, the result follows.

3. Proof of Theorem 2. Since I is a homogeneous ideal of L with L/I a free K-module, we have $L = I + J$ with J a free homogeneous K-submodule of L. Let (t_i) be an ordered, homogeneous, K-module basis of J, and let B be the set of elements of U of the form

$$t_{i_1}t_{i_2} \dots t_{i_k} \qquad (i_1 \leqslant i_2 \leqslant \cdots \leqslant i_k,\ k \geqslant 0).$$

Then, because of homogeneity, I is a free Lie algebra over K with free generating system $(\mathrm{ad}(w)x_j)_{w \in B,\ 1 \leqslant j \leqslant n}$. Moreover, every element $u \in U$ can be uniquely written in the form

$$u = \sum_{w \in B} v_w w$$

where $v_w \in V$, the enveloping algebra of I. Now suppose that

$$\mathrm{ad}(u_1)x_1 + \cdots + \mathrm{ad}(u_n)x_n = 0$$

and write

$$u_i = \sum_{j=1}^{m} v_{ij} w_j \qquad (1 \leqslant i \leqslant n)$$

with $v_{ij} \in V$ and $w_1, \dots, w_m$ distinct elements of B. Then, if $z_{ij} = \mathrm{ad}(w_j)x_i$, we have

$$\sum_{i,j} \mathrm{ad}(v_{ij})z_{ij} = 0.$$

Moreover, by introducing zero elements v_{ij} and increasing m, we can assume that the elements v_{ij} are in the subalgebra V' of V generated by the elements z_{ij}. Applying Theorem 1, we obtain that the family (v_{ij}) is a V'-linear combination of elements of the form

$$(\mathrm{ad}(v)z_{ij})ve_{ij}, \quad (\mathrm{ad}(v)z_{ij})we_{kl} + (\mathrm{ad}(w)z_{kl})ve_{ij},$$

where $v, w \in V'$ and e_{pq} is the family (d_{ij}) with $d_{ij} = 1$ if $p = i$, $q = j$ and zero otherwise. Since

$$(u_1, \dots, u_n) = \sum_{j=1}^{m} (v_{1j}, \dots, v_{nj})w_j,$$

it follows that $(u_1, \dots, u_n)$ is a V'-linear combination of elements of the form

(1) $$(\mathrm{ad}(v)z_{ij})vw_je_i = (\mathrm{ad}(vw_j)x_i)vw_je_i,$$

(2) $$(\mathrm{ad}(v)z_{ij})ww_le_k + (\mathrm{ad}(w)z_{kl})vw_je_i = (\mathrm{ad}(vw_j)x_i)ww_le_k + (\mathrm{ad}(ww_l)x_k)vw_je_i,$$

which lie in E. Q.E.D.

REFERENCES

1. N. Bourbaki, *Groupes et algèbres de Lie*, Éléments de Mathématiques, Hermann, Paris, 1972, Chapitre 2.

2. Marshall Hall, Jr., *A basis for free Lie rings and higher commutators in free groups*, Proc. Amer. Math. Soc. 1 (1950), 575–581.

3. John P. Labute, *Algèbres de Lie et pro-p-groupes définis par une seule relation*, Invent. Math. 4 (167), 142–158. MR 36 #1581.

4. ______, *The lower central series of the group* $\langle x, y \colon x^p = 1\rangle$ (to appear).

5. W. Magnus, A. Karrass and D. Solitar, *Combinatorial group theory*, Interscience, New York, 1966.

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