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SHORTER NOTES

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GROUP RINGS WHOSE UNITS FORM A NILPOTENT OR *FC* GROUP

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Let G be a finite group. We denote by RG its group ring over a ring with unity R and by $U(RG)$ its unit group. The structure of $U(RG)$ has been studied by many authors (for an excellent survey, see [**1**]). In this note, we study necessary and sufficient conditions on G for $U(RG)$ to be a nilpotent or an FC group when R is either $\mathbf{Z}$, the ring of rational integers, or a commutative ring containing $\mathbf{Z}_{(p)}$, a localization of $\mathbf{Z}$ at a prime ideal (p).

The case $R = \mathbf{Z}$ is also covered in [**8**] and either [**5**] or [**7**]; however, our proof is much simpler than the original ones, mainly because of the following result, whose proof is implicit in [**4**, p. 129].

LEMMA. *Let G be a finite group such that $TU(\mathbf{Z}G)$, the set of torsion elements in $U(\mathbf{Z}G)$, forms a subgroup. Then $TU(\mathbf{Z}G) = \pm G$, i.e. every unit of finite order is trivial.*

THEOREM 1. *Let G be a finite group. Then the following are equivalent*:
(i) *$U(\mathbf{Z}G)$ is nilpotent.*
(ii) *$U(\mathbf{Z}G)$ is an FC group.*
(iii) *$TU(\mathbf{Z}G)$ is a subgroup.*
(iv) *$TU(\mathbf{Z}G) = \pm G$.*
(v) *G is either abelian or a Hamiltonian 2-group.*

PROOF. It is well known that both (i) and (ii) imply (iii); the lemma above shows that (iii) implies (iv) and the equivalence of (iv) and (v) is also well known (see [**3**]).

Obviously, (v) implies both (i) and (ii) since for Hamiltonian 2-groups, we have that $U(\mathbf{Z}G) = \pm G$ (see [**2**, Theorem 11]).

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THEOREM 2. *Let G be a finite group and R a commutative ring containing $\mathbf{Z}_{(p)}$. Then the following are equivalent.*

(i) *$U(RG)$ is nilpotent.*

(ii) *$U(RG)$ is an FC group.*

(iii) *G is an abelian group.*

PROOF. Let J_{p^n} be the ring of integers modulo p^n. The natural epimorphism $\mathbf{Z}_{(p)} \to J_{p^n}$ induces an epimorphism $\mathbf{Z}_{(p)}G \to J_{p^n}G$ whose kernel lies in the Jacobson radical. Thus, it yields by restriction an epimorphism $U(\mathbf{Z}_{(p)}G) \to U(J_{p^n}G)$. Hence, the equivalence of (i) and (iii) follows from [**5**, Lemma 4].

Since $U(\mathbf{Z}G) \subset U(RG)$, to prove the equivalence of (ii) and (iii), it will suffice to show that if G is Hamiltonian, then $U(RG)$ is not an FC group.

A Hamiltonian group always contains a subgroup of the form

$$Q = \langle a, b | a^4 = 1, a^2 = b^2, bab^3 = a^3 \rangle,$$

i.e. isomorphic to the quaternion group of order 8. Hence, it will be enough to show that $U(\mathbf{Z}_{(p)}Q)$ is not an FC group. Since the first commutator of an FC group is torsion (see [**6**, 15.1.7]), our statement will be proved if we exhibit a commutator which is not of finite order.

Let $x, y \in Z$, $y \neq 0$, be such that $p \nmid x$ and $p | y$. Then $\alpha = x + ya$ is a unit in $\mathbf{Z}_{(p)}Q$ and

$$[b, \alpha] = b\alpha b^{-1}\alpha^{-1} = (x^2 + y^2)^{-1}(x^2 - xya + y^2a^2 + xya^3).$$

If Φ: $\mathbf{Z}_{(p)}\langle a \rangle \to \mathbf{Z}_{(p)}[i]$ is the $Z_{(p)}$-linear function such that $\Phi(a^r) = i^r$, $0 \leq r \leq 3$, then Φ is a ring homomorphism and $\Phi[b, \alpha] = X - Yi$ where

$$X = (x^2 + y^2)^{-1}(x^2 - y^2) \quad \text{and} \quad Y = (x^2 + y^2)^{-1}2xy.$$

Since X and Y are both nonzero rational numbers, $\Phi[b, \alpha]$ is not a root of unity, hence $[b, \alpha]$ is not torsion.

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